

THE DERIVE - NEWSLETTER #125

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USER GROUP

+ CAS-TI

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March 2022

A Collection of Websites which might be of Interest for You!

Operations Research

APPLICATIONS
AND ALGORITHMS
FOURTH EDITION

Wayne L. Winston
INDIANA UNIVERSITY

When I was busy with the Simplex program, I did some internet research for linear programming and other OR topics. I found a textbook which can be downloaded (more than 1400 pages). It covers a couple of interesting areas of OR and provides many exercises and review problems.

<https://itslearningakarmazyran.files.wordpress.com/2015/09/operation-research-applications-and-algorithms.pdf>

Go to the website of TI-Education and inform about the Turtle-module for Nspire-Python (and on Python's website for extended documentation)

<https://education.ti.com/en/product-resources/turtle-module/nspire-python>

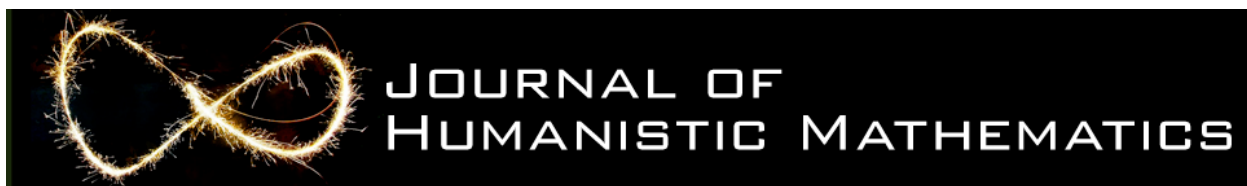
<https://education.ti.com/de/product-resources/turtle-module/nspire-python>

<https://docs.python.org/3.3/library/turtle.html?highlight=turtle>

Here you will find a lot of materials. I recommend to browse the various countries (Germany, France, Netherlands, ...)

<https://ti-unterrichtsmaterialien.net/materialien>

Robert Haas is one of the many authors in the JHM. You can browse and download numerous papers dedicated to „Humanistic Mathematics”.



<https://scholarship.claremont.edu/jhm/>

<https://scholarship.claremont.edu/jhm/vol8/iss1/15/>

<https://scholarship.claremont.edu/hmnj/>

If you are interested in differential equations and their applications then have a look to CODEE!!



Community of Ordinary Differential Equations Educators

<https://scholarship.claremont.edu/codee/>

Dear DUG-Members,

again, three months have passed and we can start the 27th year of the existence of DUG.

This issue has become particularly extensive with 52 pages. Responsible for this are several graphics and screen shots, which I believe are valuable for the articles - and besides, they break up the text.

As always, I would like to draw your attention to a few items in particular: first of all, there is the extensive list of links to interesting websites.

I came across the Journal for Humanistic Mathematics in my communication with Dr. Robert Haas. He has published several papers in this journal. His generalization of the conic section definition is also there (p. 18). A further question is raised, the solution of which is still open (p. 32). Look forward to his "cello problem" in the next DNL.

If someone is looking for applications of differential equations, then he is well served by the CODEE Journal. Several issues of both journals can be downloaded without of charge (pdf files).

Rick Nungester's Spirograph article brings back memories of the Maurer Roses from DNL#7, just as the SIMPLEX article may remind you of Bruce Chaffee in DNL#26.

The 4th part of Michel Beaudin's large contribution deals with differential equations. The translations into German and English are included in the MTH125.ZIP file.

I would not like to forget the news of our friend David Halprin, who has made a considerable part of his extensive publications available on the Internet (User Forum),

Until the next time I remain

Yours Josef

Liebe DUG-Mitglieder,

wiederum sind drei Monate vergangen und wir können das 27. Jahr des Bestehens der DUG beginnen.

Diese Ausgabe ist mit 52 Seiten besonders umfangreich geworden. Verantwortlich dafür sind mehrere Grafiken und screen shots, von denen ich glaube, dass sie wertvoll für die Beiträge sind – und außerdem lockern sie den Text auf.

Ich möchte Euch wie immer auf einige Inhalte besonders hinweisen: da ist zuerst die umfangreiche Liste von Links zu interessanten websites

Auf das Journal for Humanistic Mathematics bin ich in meiner Kommunikation mit Dr. Robert Haas gestoßen. Er hat auch schon einiges in diesem Journal veröffentlicht. Seine Verallgemeinerung der Kegelschnittsdefinition findet sich auch darunter. Eine weiterführende Frage wird angeschnitten, dessen Lösung noch offen ist (Seite 32). Freuen Sie sich auf sein „Cello-Problem“ im nächsten DNL

Wenn jemand Anwendungen der Differentialgleichungen sucht, dann ist er mit dem CODEE Journal bestens bedient. Von beiden Journalen können ohne Gebühr mehrere Jahrgänge heruntergeladen werden (pdf-Dateien).

Rick Nungesters Spirograph-Beitrag ruft Erinnerung an die Maurer Roses vom DNL#7 wach, genauso wie der SIMPLEX-Artikel an Bruce Chaffee in DNL#26 erinnern kann.

Der 4. Teil von Michel Beaudins großem Beitrag beschäftigt sich mit Differentialgleichungen. Die Übersetzungen ins Deutsche und Englische sind in der MTH125.ZIP-Datei enthalten.

Nicht vergessen möchte ich die Nachricht unseres Freundes David Halprin, der einen beträchtlichen Teil seiner umfangreichen Publikationen im Internet für alle zugänglich gemacht hat (User Forum).

Bis zum nächsten Mal verbleibe ich

Euer Josef

The *DERIVE-NEWSLETTER* is the Bulletin of the *DERIVE & CAS-TI User Group*. It is published at least four times a year with a content of 40 pages minimum. The goals of the *DNL* are to enable the exchange of experiences made with *DERIVE*, *TI-CAS* and other CAS as well to create a group to discuss the possibilities of new methodical and didactical manners in teaching mathematics.

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Contributions:

Please send all contributions to the Editor. Non-English speakers are encouraged to write their contributions in English to reinforce the international touch of the *DNL*. It must be said, though, that non-English articles will be warmly welcomed nonetheless. Your contributions will be edited but not assessed. By submitting articles, the author gives his consent for reprinting it in the *DNL*. The more contributions you will send, the more lively and richer in contents the *DERIVE & CAS-TI Newsletter* will be.

Next issue:

June 2022

Preview: Contributions waiting to be published

Some simulations of Random Experiments, J. Böhm, AUT, Lorenz Kopp, GER
Wonderful World of Pedal Curves, J. Böhm, AUT
Simulating a Graphing Calculator in *DERIVE*, J. Böhm, AUT
Cubics, Quartics – Interesting features, T. Koller & J. Böhm, AUT
Logos of Companies as an Inspiration for Math Teaching
Exciting Surfaces in the FAZ, BooleanPlots.mth, P. Schofield, UK
Old traditional examples for a CAS – What's new? J. Böhm, AUT
Mandelbrot and Newton with *DERIVE*, Roman Hašek, CZ
Tutorials for the NSpireCAS, G. Herweyers, BEL
Dirac Algebra, Clifford Algebra, Vector-Matrix-Extension, D. R. Lunsford, USA
Another Approach to Taylor Series, D. Oertel, GER
Charge in a Magnetic Field, H. Ludwig, GER
More Applications of TI-Innovator™ Hub and TI-Innovator™ Rover
Surfaces and their Duals, Cayley Symmetroid, J. Böhm, AUT
A Collection of Special Problems, Hemme's problems, W. Alvermann, GER
DERIVE Bugs? D. Welz, GER
Tweening & Morphing with TI-NspireCX-II-T, J. Böhm, AUT
The Gap between Poor and Rich, J. Böhm, AUT
More functions from M. Myers and from Bhuvanesh's Mathtools-library
TaxiCab Conics, R. Haas, USA
QR-Code light, Random numbers following a given distribution
Polygons and their properties, Quartiles, F. de Jesús Martínez Vargas, Mexico
Penalty Shootout mathematics, B. Grabinger, GER
Sparse Matrices, 153 is another Special Number, and others

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Résumé 1 – Part 4

Résolution symbolique, résolution numérique d'équations et de systèmes d'équations Rappels et compléments en équations différentielles

3- Équations différentielles : rappels et compléments

Si l'on a fait son BACC à l'ÉTS, aucune preuve du théorème d'existence et d'unicité n'était donnée. Nous en parlons ici puisque la preuve utilise encore l'idée d'un « point » fixe.

3.1 Théorème Soit le problème $\frac{dy}{dx} = f(x, y)$, $y(x_0) = y_0$. Supposons que f ainsi que $\frac{\partial f}{\partial y}$ soient continues dans un certain rectangle $R = \{(x, y) : a \leq x \leq b, c \leq y \leq d\}$ contenant le point (x_0, y_0) . Alors il existe un nombre positif h et une unique fonction $\varphi : [x_0 - h, x_0 + h] \rightarrow R$ tels que

$$\begin{aligned}\varphi(x_0) &= y_0, \\ \frac{d\varphi}{dx} &= f(x, \varphi(x)), \quad x_0 - h < x < x_0 + h.\end{aligned}$$

La preuve de ce théorème (généralisable aux systèmes d'équations différentielles : voir théorème 3.3) consistera à démontrer que la suite de fonctions, définie par $y_{n+1} = T y_n$, $y_0(x) = y_0$, converge (uniformément) sur $x_0 - h, x_0 + h$ vers cette solution. Ici, T sera l'opérateur (non-linéaire) définie par

$$T[y](x) = y_0 + \int_{x_0}^x f(t, y(t)) dt$$

La suite $y_{n+1} = y_0 + \int_{x_0}^x f(t, y_n(t)) dt$, $n = 0, 1, 2, \dots$ est appelée suite des itérations de Picard.

Preuve: posons donc $y_{n+1} = y_0 + \int_{x_0}^x f(t, y_n(t)) dt$, $n = 0, 1, 2, \dots$. On remarque que

$y_n(x) = y_0(x) + \sum_{k=1}^n (y_k(x) - y_{k-1}(x))$. Ainsi, la suite $\{y_n(x)\}_{n=1}^{\infty}$ converge si et seulement si la série

$\sum_{k=1}^n (y_k(x) - y_{k-1}(x))$ converge. Voici les étapes:

a) f et $\frac{\partial f}{\partial y}$ sont bornées sur R puisque continues. Alors, il existe des constantes M et K telles que

$$|f(x, y)| \leq M, \quad \left| \frac{\partial f}{\partial y} \right| \leq K \quad \forall (x, y) \in R$$

Par le théorème de la valeur moyenne, on peut écrire que

$$|f(x, y_1) - f(x, y_2)| = \left| \frac{\partial f}{\partial y}(x, y^*) \right| |y_1 - y_2| \leq K |y_1 - y_2|$$

pour tout couple de points $(x, y_1), (x, y_2) \in R$. On peut donc choisir h positif tel que $Kh < 1$ et tel que le rectangle $R' = \{(x, y) : |x - x_0| \leq h, |y - y_0| \leq Mh\} \subset R$.

b) Montrons que la série converge : on remarque que chacune des fonctions $y_n(x)$ possède un graphe contenu dans R' . En effet, c'est clair pour $y_0(x)$ et

$$|y_1(x) - y_0| = \left| \int_{x_0}^x f(t, y_0(t)) dt \right| \leq M h.$$

Ainsi, les points $(t, y_1(t)) \in R'$, donc $|f(t, y_1(t))| \leq M$ et alors

$$|y_2(x) - y_0| = \left| \int_{x_0}^x f(t, y_1(t)) dt \right| \leq M h$$

et ainsi de suite. La fonction $y_1(x)$ est continue sur l'intervalle fermé $|x - x_0| \leq h$, donc possède un maximum. Posons alors $a = \max |y_1(x) - y_0|$. Ainsi

$$|f(t, y_1(t)) - f(t, y_0(t))| \leq K |y_1(t) - y_0(t)| \leq K a,$$

d'où $|y_2(x) - y_1(x)| = \left| \int_{x_0}^x (f(t, y_1(t)) - f(t, y_0(t))) dt \right| \leq K a h = a(K h)$. De la même manière,

$$|f(t, y_2(t)) - f(t, y_1(t))| \leq K |y_2(t) - y_1(t)| \leq K^2 a h,$$

d'où $|y_3(x) - y_2(x)| = \left| \int_{x_0}^x (f(t, y_2(t)) - f(t, y_1(t))) dt \right| \leq (K^2 a h) h = a(K h)^2$. Mais alors,

$$|y_n(x) - y_{n-1}(x)| \leq a(K h)^{n-1}$$

et notre série converge par comparaison avec la série géométrique! Dénotons sa somme $y(x)$:

$$y(x) = y_0(x) + \sum_{n=1}^{\infty} (y_n(x) - y_{n-1}(x)).$$

Remarquons que la convergence est même uniforme et donc $y(x)$ est une fonction continue avec un graphe dans R' .

c) Montrons que $y(x)$ satisfait bien l'É.D. La condition initiale est trivialement satisfaite et on va montrer que

$$y(x) - y_0 - \int_{x_0}^x f(t, y(t)) dt = 0.$$

On sait que $y_n(x) - y_0 - \int_{x_0}^x f(t, y_{n-1}(t)) dt = 0$. Alors

$$y(x) - y_0 - \int_{x_0}^x f(t, y(t)) dt = y(x) - y_n(x) + \int_{x_0}^x (f(t, y_{n-1}(t)) - f(t, y(t))) dt$$

et donc

$$\left| y(x) - y_0 - \int_{x_0}^x f(t, y(t)) dt \right| \leq |y(x) - y_n(x)| + \left| \int_{x_0}^x (f(t, y_{n-1}(t)) - f(t, y(t))) dt \right|$$

$$\leq |y(x) - y_n(x)| + Kh \max |y_{n-1}(x) - y(x)|$$

puisque le graphe de $y(x)$ est dans R' et donc dans R . La convergence uniforme fait en sorte que l'on peut rendre aussi petit que l'on veut la dernière somme, en choisissant n suffisamment grand.

d) L'unicité. On argumente de façon semblable à la preuve du théorème 2.1 puisque le même principe de contraction a été utilisé. ♦

3.2 Exemples

3.2.1 Pour l'É.D. $y' = -y + 2x + 1$, $y(0) = 1$, la suite des itérations de Picard donne exactement le développement de Taylor de la solution $2e^{-x} + 2x - 1$. Remarquons que cette solution est définie sur toute la droite. Avant même de résoudre l'É.D., on était assuré de l'existence et de l'unicité d'une solution au voisinage du point $(0, 1)$ puisque la fonction $f(x, y) = -y + 2x + 1$ ainsi que sa dérivée partielle par rapport à y sont continues dans un rectangle (aussi grand soit-il) englobant le point $(0, 1)$.

3.2.2 Pour l'É.D. $y' = 1 + y^2$, $y(0) = 0$, la suite des itérations de Picard va « ressembler » au développement de Taylor de la solution $y = \tan x$. On voit ici que, bien que f et $\frac{\partial f}{\partial y}$ soient continues dans tout rectangle contenant l'origine (aussi grand soit-il), on ne peut étendre la solution au-delà de $\pi/2$. Cela ne se produit pas pour une équation linéaire (c'était une É.D. linéaire à l'exemple 3.2.1).

3.2.3 Pour l'É.D. $y' = 3y^{2/3}$, $y(2) = 0$, le théorème ne s'applique pas. On trouve d'ailleurs deux solutions : $y(x) \equiv 0$ (la fonction identiquement nulle) et $y(x) = (x - 2)^3$.

3.2.4 Même chose pour l'É.D. $y' = \sqrt{y}$, $y(0) = 0$. On trouve 2 solutions : $y(x) = 0$ (la fonction identiquement nulle) et

$$y(x) = \begin{cases} 0 & \text{si } x < 0 \\ \frac{x^2}{4} & \text{si } x \geq 0 \end{cases}.$$

3.2.5 Exemple L'unique solution prédite par le théorème d'existence et d'unicité peut très bien faire intervenir des nombres complexes. Aussi, dans les cas où il sera possible d'identifier explicitement la solution unique, il sera recommandé d'en indiquer le domaine. Voici un exemple. Considérons l'É.D. (« séparable » mais aussi « linéaire » : voir le rappel en 3.5 plus loin)

$$\frac{dy}{dx} = \frac{4y}{x^2 - 9}, y(0) = 5$$

Le théorème d'existence et d'unicité s'applique puisque la fonction $f(x, y) = \frac{4y}{x^2 - 9}$ ainsi que sa dérivée partielle $\frac{\partial f}{\partial y} = \frac{4}{x^2 - 9}$ sont continues dans un voisinage de point $(0, 5)$. Il est toutefois évident que le domaine sera un sous-ensemble de l'intervalle $-3 < x < 3$ ici.

3.2.6 Exemple L'unique solution prédite par le théorème d'existence et d'unicité ne pourra pas toujours être trouvée explicitement. Et lorsque c'est possible, il est important de bien spécifier l'intervalle d'existence de la solution. Prenons l'exemple suivant, une É.D. séparable mais non linéaire et facile à résoudre à la main : $\frac{dy}{dx} = y^3$, $y(0) = 1$. Évidemment, les hypothèses du théorème 3.1 sont satisfaites puisque les fonctions y^3 et $3y^2$ sont continues dans un voisinage du point $(0, 1)$. Notons que ces deux fonctions ont beau être continues dans tout le plan, il n'est pas dit que l'intervalle d'existence de la solution qu'on va trouver est aussi grand qu'on veut! On sépare les variables et intègre de chaque côté :

$$\int_1^y \frac{1}{s^3} ds = \int_0^x 1 dt \Rightarrow \frac{1}{2} - \frac{1}{2y^2} = x$$

Heureusement, il est possible de résoudre ici pour la variable « y » et on trouve, après résolution de l'équation quadratique en « y » qu'on doit conserver la solution

$$y = \frac{1}{\sqrt{1-2x}}, x < \frac{1}{2}.$$

On peut vérifier que la fonction $\varphi(x) = \frac{1}{\sqrt{1-2x}}$, $x < \frac{1}{2}$ satisfait bien l'É.D. et la condition initiale puisque

$$\varphi(0) = \frac{1}{\sqrt{1}} \text{ et } \frac{d\varphi}{dx} - \varphi(x)^3 = \frac{1}{(1-2x)^{3/2}} - \frac{1}{(1-2x)^{3/2}} = 0.$$

Dans le résumé 2, nous parlerons de systèmes d'É.D. du premier ordre et des résultats comme ceux des deux prochains théorèmes seront annoncés. Les preuves ne sont pas données ici.

3.3 Théorème (Itérations de Picard pour les systèmes d'équations différentielles). Supposons que \mathbf{f} et $\frac{\partial \mathbf{f}}{\partial x_i}$ ($i = 1, 2, \dots, n$) soient continues dans l'ensemble ouvert suivant

$$R = \{(t, x_1, x_2, \dots, x_n) : a < t < b, c_i < x_i < d_i, i = 1, \dots, n\},$$

ensemble contenant le point (t, \mathbf{x}_0) où $\mathbf{x}_0 = x_{1,0} \ x_{2,0} \ \dots \ x_{n,0}^T$. Alors le problème

$$\mathbf{x}'(t) = \mathbf{f}(t, \mathbf{x}), \quad \mathbf{x}(t_0) = \mathbf{x}_0$$

admet une solution unique dans un certain intervalle $t_0 - h, t_0 + h$.

3.4 Théorème Soient $\mathbf{A}(t)$ une matrice carrée d'ordre n et $\mathbf{f}(t)$ une fonction vectorielle toutes deux continues sur un intervalle ouvert $a < t < b$ contenant le point t_0 . Alors, pour tout vecteur \mathbf{x}_0 , il existe une unique solution au problème

$$\mathbf{x}'(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{f}(t), \quad \mathbf{x}(t_0) = \mathbf{x}_0$$

et cette solution est définie sur tout l'intervalle $a < t < b$.

3.5 Rappel L'équation linéaire du premier ordre

$$\frac{dy}{dx} + p(x)y = q(x),$$

possède la solution générale

$$y = \frac{1}{\mu(x)} \left(\int \mu(x)q(x)dx + C \right)$$

où $\mu(x) = e^{\int p(x)dx}$. Rappelons d'où sort cette formule. On voudrait que le membre de gauche de l'É.D. soit la dérivée d'un produit. Soit donc une fonction inconnue $\mu(x)$ et considérons l'É.D.

$$\mu(x)y' + p(x)y\mu(x) = \mu(x)q(x).$$

Puisque $\frac{d}{dx}(\mu y) = \mu y' + y\mu'$, alors choisissons μ telle que $\mu' = \mu p$ i.e. soit $\mu(x) = e^{\int p(x)dx}$.

Alors, on obtient la formule annoncée ci-haut. S'il y a une condition initiale du genre $y(x_0) = y_0$, alors on continue comme suit : on a choisi μ comme ci-haut et donc

$$\frac{d}{dx}(\mu y) = \mu q.$$

On intègre ce chaque côté entre x_0 et x pour obtenir la formule

$$y = \frac{1}{\mu(x)} \left(\int_{x_0}^x \mu(s)q(s)ds + y_0\mu(x_0) \right)$$

3.6 Exemples Équations différentielles linéaires du premier ordre ou systèmes linéaires du premier ordre facilement résolubles :

3.6.1 Quel temps met un parachutiste pour atteindre le sol si sa masse totale est de 75 kg, qu'il saute d'une hauteur de 4000 m et si l'on suppose que la force de résistance de l'air est proportionnelle à la vitesse avec constante de proportionnalité de 15 kg/s pour les 60 premières secondes de chute (avant l'ouverture du parachute) et de 105 kg/s pour le reste de la chute ? (réponse: 241.56 secondes en prenant $g = 9.81 \text{ m/s}^2$).

3.6.2 Deux gros réservoirs, contenant chacun 50 litres (L) de liquide, sont reliés par un tuyau et le liquide coule du réservoir A vers le réservoir B à un taux de 5 L/min. On maintient le liquide dans chaque réservoir uniforme par brassage. Une solution saline de concentration (en sel) de 3 kg/L est versé dans le réservoir A au taux de 5 L/min. La solution s'échappe du réservoir B au taux de 5L/min. Sachant qu'il y a initialement 50 kg de sel dans le réservoir A et 100 kg de sel dans le réservoir B, il faut trouver la masse de sel dans chacun des réservoirs pour tout $t \geq 0$.

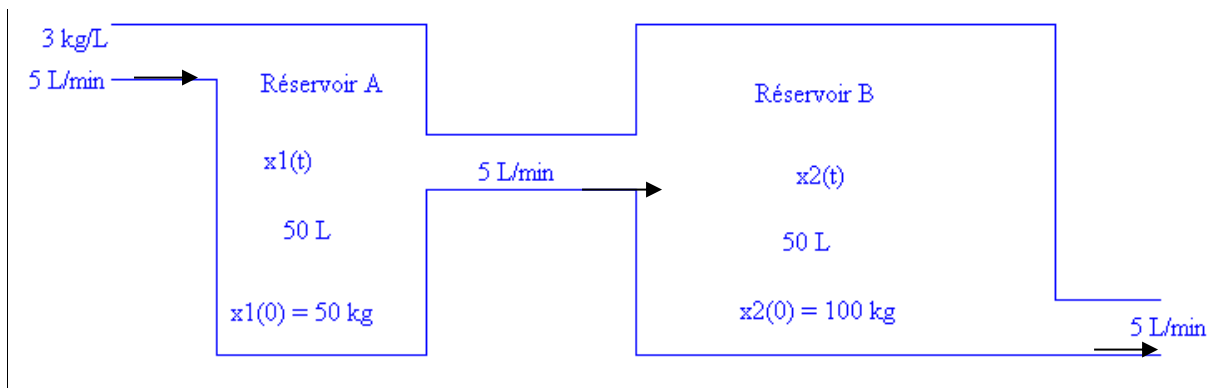


Figure 3.1

Vérifiez que la réponse est $150 - 100 \cdot e^{-t/10}$ pour A et $150 - (10t + 50) \cdot e^{-t/10}$ pour B.

Puisque nous étudierons les systèmes d'équations différentielles, rappelons certains résultats pour le cas de l'É.D. linéaire d'ordre 2 plutôt que d'ordre « n ». Donc soit

$$\frac{d^2 y}{dx^2} + p(x) \frac{dy}{dx} + q(x)y = r(x)$$

où les fonction p , q et r sont continues sur un certain intervalle $a < x < b$.

3.7 Théorème Supposons que les fonctions p , q et r sont continues sur l'intervalle $]a, b[$ et que x_0 soit un point de cet intervalle. Alors le problème

$$\frac{d^2 y}{dx^2} + p(x) \frac{dy}{dx} + q(x)y = r(x), y(x_0) = y_0, y'(x_0) = v_0$$

admet une solution unique $y(x)$ sur cet intervalle.

L'unicité de la solution peut être démontrée en appliquant les théorèmes 3.3 et 3.4 au système suivant, système équivalent à l'É.D. (on pose $y' = z$)

$$\begin{cases} y' = z & y(x_0) = y_0 \\ z' = -p(x)z - q(x)y + r(x) & z(x_0) = v_0 \end{cases}$$

Contrairement à l'équation linéaire du premier ordre où une formule close donnant la solution existe, on doit, ici, procéder comme suit pour trouver la solution de notre problème. On trouve d'abord 2 fonctions y_1 et y_2 satisfaisant (l'équation « complémentaire » ou « homogène »)

$$y'' + p(x)y' + q(x)y = 0$$

et $y_1(c)y_2'(c) - y_1'(c)y_2(c) \neq 0$ en un point c de l'intervalle $]a, b[$. On trouve ensuite une solution particulière $\phi(x)$. La solution au problème est alors

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \phi(x)$$

avec c_1, c_2 calculés et trouvés par le fait que le déterminant (appelé le Wronskien de y_1, y_2)

$$W_{y_1, y_2}(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$$

est différent de zéro : on dit alors que la paire de solutions y_1, y_2 forment un ensemble fondamental de solutions ou qu'elles sont linéairement indépendantes sur $]a, b[$.

Si l'on ne peut trouver qu'une seule solution y_1 , on obtient une seconde solution y_2 , indépendante de y_1 par la méthode de réduction de l'ordre, ce qui donne

$$y_2 = v y_1 \text{ où } v' = \frac{e^{-\int p(x) dx}}{y_1^2}.$$

Si l'on ne peut même pas trouver une solution, on peut procéder par série de puissances et cette méthode permet d'introduire certaines fonctions spéciales. La solution particulière se trouve par la méthode des coefficients indéterminés dans le cas où l'É.D. est à coefficients constants et où $r(x)$ est formé de sommes et produits de polynômes, d'exponentielles et de sinus et cosinus. Dans le cas général, une solution particulière $\phi(x)$ peut toujours s'obtenir par la méthode de variation des paramètres, ce qui donne avec W défini précédemment

$$\phi(x) = -y_1(x) \int \frac{y_2(x)r(x)}{W} dx + y_2(x) \int \frac{y_1(x)r(x)}{W} dx.$$

3.8 Rappel Soient a, b et c 3 nombres réels, $a \neq 0$. Alors la solution générale de

$$a y'' + b y' + c y = 0$$

dépend du discriminant $\Delta = b^2 - 4ac$ et des racines de l'équation caractéristique associée

$$a \lambda^2 + b \lambda + c = 0.$$

3.8.1 Si $\Delta > 0$, alors la solution générale de l'É.D. est $y = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$ où λ_1, λ_2 sont les 2 racines réelles et distinctes de l'équation caractéristique.

3.8.2 Si $\Delta = 0$, alors la solution générale est $y = e^{\lambda t} (c_1 + c_2 t)$ où λ est la racine double de l'équation caractéristique.

3.8.3 Si $\Delta < 0$, alors la solution générale est $y = e^{\alpha t} (c_1 \cos(\omega t) + c_2 \sin(\omega t))$ où α, ω sont respectivement les parties réelle et imaginaire des solutions de l'équation caractéristique.

3.9 Exemples

3.9.1 L'amplitude de la solution en régime permanent dans le cas d'un mouvement harmonique amorti forcé avec force sinusoïdale. On va considérer un problème de masse-ressort. Dans l'É.D. qui suit, m, b, k, F sont des constantes positives fixées et $b^2 < 4mk$.

$$m y'' + b y' + k y = F \cos(\omega t)$$

Peu importe le signe de $b^2 < 4mk$ et les conditions initiales imposées, la solution complémentaire va tendre vers 0 lorsque t tend vers l'infini (car $b > 0$). Puisque $b \neq 0$, le candidat pour la solution particulière —appelée ici solution en régime permanent— est nécessairement de la forme suivante : $A \cos(\omega t) + B \sin(\omega t)$. On verra que, dans le cas du mouvement sous-amorti ($b^2 < 4mk$), l'amplitude de la solution en régime permanent est

$$M(\omega) = \frac{F/m}{\sqrt{\left(\frac{k}{m} - \omega^2\right)^2 + \left(\frac{b}{m}\right)^2 \omega^2}}$$

et que cette fonction atteint son maximum au point

$$\omega = \omega_r = \sqrt{\frac{k}{m} - \frac{b^2}{2m^2}} \quad \text{lorsque } b^2 < 2mk.$$

3.9.2 Voici un cas particulier de l'exemple 3.9.1. On considère le problème de masse-ressort suivant :

$$y'' + \frac{1}{4} y' + 2y = 2 \cos(\omega t), \quad y(0) = 0, \quad y'(0) = 2 \quad [\omega > 0].$$

On tracera le graphique de l'amplitude du régime permanent en fonction de ω et cherchera son maximum. On s'attend à un maximum atteint en $\omega = \frac{3\sqrt{14}}{8} \approx 1.40312$ dont la valeur est $\frac{64\sqrt{127}}{127} \approx 5.67908$. Nous vérifierons cela.

3.9.3 Remarquons que l'É.D. $ay'' + by' + cy = 0$ est équivalente au système du premier ordre

$$\begin{cases} y' = z \\ z' = -\frac{b}{a}z - \frac{c}{a}y \end{cases} \text{ ou, en notation matricielle, } \begin{bmatrix} y' \\ z' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix}$$

Nous verrons que les « valeurs propres » de la matrice carrée précédente ne sont rien d'autres que les racines de l'équation caractéristique $a\lambda^2 + b\lambda + c = 0$!

3.9.4 MHS ans le cas du mouvement harmonique simple, $my'' + ky = 0$, on a en multipliant par y' ,

$$my' y'' + ky' y = 0,$$

d'où par intégration $\frac{1}{2}mv^2 + \frac{1}{2}ky^2 = E_0 = \text{constante}$ ($v = y'$ est la vitesse). Le premier terme est l'énergie cinétique, le second terme est l'énergie potentielle et E_0 est l'énergie totale. Mais nous savons que la solution du MHS est

$$y = a_1 \cos(\omega_0 t) + b_1 \sin(\omega_0 t) = c_1 e^{i\omega_0 t} + c_{-1} e^{-i\omega_0 t}$$

avec $\omega_0^2 = \frac{k}{m}$, $c_1 = \frac{a_1 - ib_1}{2}$, $c_{-1} = \bar{c}_1$. Écrivons que $A = c_1 e^{i\omega_0 t}$, $B = c_{-1} e^{-i\omega_0 t}$. Alors $y = A + B$ et en dérivant, on obtient $v = y' = A' + B' = i\omega_0(A - B)$. Mais alors, un calcul direct donne

$$\begin{aligned} E_0 &= \frac{1}{2}m(i\omega_0(A - B))^2 + \frac{1}{2}k(A + B)^2 = \\ &= \frac{1}{2}m\omega_0^2(-A^2 + 2AB - B^2 + A^2 + 2AB + B^2) = \\ &= 2kAB = 2k|c_1|^2 \end{aligned}$$

L'énergie est proportionnelle au carré de l'amplitude $|c_1|$: $E_0 = 2k|c_1|^2$. Cet important résultat reviendra dans l'analyse de Fourier.

3.9.5 Exemple La « bonne vieille » méthode des coefficients indéterminés (et l'utilisation des nombres complexes) est très utile. Trouvons la solution en régime permanent du problème

$$y'' + 6y' + 13y = 3\cos(2t), \quad y(0) = 0, \quad y'(0) = 0.$$

La solution particulière n'est rien d'autre que la partie réelle de la solution particulière complexe au problème $y'' + 6y' + 13y = 3e^{2it}$. Un calcul direct montre que, essayant le candidat complexe Ae^{2it} ,

on trouve $A = \frac{3}{9+12i} = \frac{3}{25} - \frac{4}{25}i$. Tous ces calculs sont automatisables encore facilement : on n'oublie d'informer Nspire que les variables sont complexes et non réelles (d'où l'ajout du « $_$ » au début des calculs) : on vérifie ensuite si la solution particulière trouvée est bien bonne.

The screenshot shows the following steps in a CAS environment:

- Define $can := a_{_} \cdot e^{2 \cdot i \cdot t_{_}}$
- Compute the derivative: $\frac{d^2}{dt^2}(can) + 6 \cdot \frac{d}{dt}(can) + 13 \cdot can$
- Solve for $a_{_}$: $cSolve(a_{_} \cdot (9 + 12 \cdot i) = 3, a_{_})$
- Take the real part: $real\left(\left(\frac{3}{25} - \frac{4}{25} \cdot i\right) \cdot e^{2 \cdot i \cdot t}\right)$
- Define the particular solution: $sp := \frac{3 \cdot \cos(2 \cdot t)}{25} + \frac{4 \cdot \sin(2 \cdot t)}{25}$
- Verify the solution: $\frac{d^2}{dt^2}(sp) + 6 \cdot \frac{d}{dt}(sp) + 13 \cdot sp$

Figure 3.2

Solution of example 3.6.1 (Josef):

The screenshot shows the following steps in a CAS environment:

- The Parachutist**
- Parachutist**
- $m=75, k1=15, k2=105$
- $m \cdot v1'(t) = g \cdot m - k1 \cdot v1(t), v1(0)=0; y1'(t) = v1(t), y1(0)=0$
- $m \cdot v2'(t) = g \cdot m - k2 \cdot v2(t), v2(0) = v1(60); y2'(t) = v2(t), y2(0) = y1(60)$
- $y2(t) = 4000; t = ?; \text{total time} = 60 + t$
- $m:=75:k1:=15:k2:=105:g:=9.81 \rightarrow 9.81$
- $deSolve(m \cdot v1' = g \cdot m - k1 \cdot v1 \text{ and } v1(0)=0, t, v1) \rightarrow v1 = 49.05 - 49.05 \cdot (0.818731)^t$
- $v1 := 49.05 - 49.05 \cdot (0.818731)^t |_{t=60} \rightarrow 49.0497 = \text{velocity after 60 seconds}$
- $deSolve(y1' = 49.05 - 49.05 \cdot (0.818731)^t \text{ and } y1(0)=0, t, y1) \rightarrow y1 = 245.25 \cdot (0.818731)^t + 49.05 \cdot t - 245.25$
- $y1 := 245.25 \cdot (0.818731)^t + 49.05 \cdot t - 245.25 |_{t=60} \rightarrow 2697.75 = \text{fallen height after 60 seconds}$
- $deSolve(m \cdot v2' = g \cdot m - k2 \cdot v2 \text{ and } v2(0) = v1, t, v2) \rightarrow v2 = 42.0426 \cdot (0.246597)^t + 7.00714$
- $v2 := 42.0426 \cdot (0.246597)^t + 7.00714 \rightarrow 42.0426 \cdot (0.246597)^t + 7.00714$
- $deSolve(y2' = v2 \text{ and } y2(0) = y1, t, y2) \rightarrow y2 = -30.0304 \cdot (0.246597)^t + 7.00714 \cdot t + 2727.78$
- $y2 := -30.0304 \cdot (0.246597)^t + 7.00714 \cdot t + 2727.78 \rightarrow -30.0304 \cdot (0.246597)^t + 7.00714 \cdot t + 2727.78$
- $60 + nSolve(y2 = 4000, t) \rightarrow 241.561 = \text{total time}$

Solution of example 3.6.2 (Josef):

Two Tanks

$\Delta x = \text{amount input} - \text{amount output}$
 $= (\text{concentration}_{in} \times \text{rate}_{in}) \times \Delta t - (\text{concentration}_{out} \times \text{rate}_{out}) \times \Delta t$

For A:
 $dx_1(t)/dt = 3 \cdot 5 - 5x_1(t)/50 \cdot 5, \quad x_1(0) = 50$

$\text{deSolve}\left(x_1' = 15 - \frac{5 \cdot x_1}{50} \text{ and } x_1(0) = 50, t, x_1\right) \rightarrow x_1 = 150 - 100 \cdot e^{-\frac{t}{10}}$

$x_1 := 150 - 100 \cdot e^{-\frac{t}{10}} \rightarrow 150 - 100 \cdot e^{-\frac{t}{10}}$

For B:
 $dx_2(t)/dt = x_1(t)/50 \cdot 5 - x_2(t)/50 \cdot 5, \quad x_2(0) = 100$

$\text{deSolve}\left(x_2' = \frac{x_1}{50} \cdot 5 - \frac{x_2}{50} \cdot 5 \text{ and } x_2(0) = 100, t, x_2\right) \rightarrow x_2 = 10 \cdot e^{-\frac{t}{10}} \cdot \left(15 \cdot e^{\frac{t}{10}} - t - 5\right) = 150 - 10 \cdot e^{-0.1t} (10t + 50)$

Solution of example 3.6.2 in the *DERIVE*-way (Josef):

$$\begin{aligned} \#1: & \int \frac{1}{15 - 0.1 \cdot x} dx = \int 1 dt + c \\ \#2: & -10 \cdot \text{LN}(x - 150) = c + t \\ \#3: & \text{SOLVE}(-10 \cdot \text{LN}(x - 150) = c + t, x) \\ \#4: & x = e^{-\frac{c}{10} - \frac{t}{10}} + 150 \\ \#5: & x = c \cdot e^{-\frac{t}{10}} + 150 \\ \#6: & \text{SOLVE}(50 = c \cdot e^{-\frac{0}{10}} + 150, c) = (c = -100) \\ \#7: & x = 150 - 100 \cdot e^{-\frac{t}{10}} \\ \#8: & \text{DSOLVE1}(0.1 \cdot y - 15 - 10 \cdot e^{-0.1 \cdot t}, 1, t, y, 0, 100) \\ \#9: & e^{\frac{t}{10}} \cdot (y - 150) - 10 \cdot t = -50 \\ \#10: & \text{SOLVE}(e^{\frac{t}{10}} \cdot (y - 150) - 10 \cdot t = -50, y) \\ \#11: & y = e^{-\frac{t}{10}} \cdot (10 \cdot t - 50) + 150 \end{aligned}$$

Solution of example 3.9.1 (Josef)

$$y'' + \frac{1}{4}y' + 2y = 2 \cos(\omega t), \quad y(0) = 0, \quad y'(0) = 2 \quad [\omega > 0]$$

Solving the homogeneous differential equation:

$$\text{characteristic equation: } cZeros\left(s^2 + \frac{s}{4} + 2, s\right) \rightarrow \left\{ \frac{-1}{8} + \frac{\sqrt{127}}{8} \cdot i, \frac{-1}{8} - \frac{\sqrt{127}}{8} \cdot i \right\}$$

$$\text{complementary solution: } yc(t) := e^{\frac{-t}{8}} \cdot \left(a \cdot \cos\left(\frac{\sqrt{127}}{8} \cdot t\right) + b \cdot \sin\left(\frac{\sqrt{127}}{8} \cdot t\right) \right) \rightarrow \text{Fertig}$$

$$yc1(t) := \frac{d}{dt}(y(t)) \rightarrow \text{Fertig}$$

$$yc1(t)$$

$$\rightarrow \frac{8 \cdot \omega^2 \cdot \cos(\omega \cdot t)}{16 \cdot \omega^4 - 63 \cdot \omega^2 + 64} + \frac{32 \cdot \omega \cdot (\omega^2 - 2) \cdot \sin(\omega \cdot t)}{16 \cdot \omega^4 - 63 \cdot \omega^2 + 64} + 2 \cdot e^{\frac{-t}{8}} \cdot \cos\left(\frac{\sqrt{127}}{8} \cdot t\right) - \frac{2 \cdot \sqrt{127} \cdot e^{\frac{-t}{8}} \cdot \sin\left(\frac{\sqrt{127}}{8} \cdot t\right)}{127}$$

$$\text{solve}(yc(0)=0 \text{ and } yc1(0)=2, \{a, b\}) \rightarrow a=0 \text{ and } b=1 \text{ and } \omega=0$$

$$ycc(t) := \frac{e^{\frac{-t}{8}} \cdot 16 \cdot \sqrt{127}}{127} \cdot \sin\left(\frac{\sqrt{127}}{8} \cdot t\right) \rightarrow \text{Fertig}$$

$$\text{Check the solution: } \frac{d^2}{dt^2}(ycc(t)) + \frac{1}{4} \cdot \frac{d}{dt}(ycc(t)) + 2 \cdot ycc(t) \rightarrow 0$$

particular solution applying variation of parameters:

$$yp(t) := \frac{-32 \cdot (\omega^2 - 2)}{16 \cdot \omega^4 - 63 \cdot \omega^2 + 64} \cdot \cos(\omega \cdot t) + \frac{8 \cdot \omega}{16 \cdot \omega^4 - 63 \cdot \omega^2 + 64} \cdot \sin(\omega \cdot t) \rightarrow \text{Fertig}$$

$$\frac{d^2}{dt^2}(yp(t)) + \frac{1}{4} \cdot \frac{d}{dt}(yp(t)) + 2 \cdot yp(t) = 2 \cdot \cos(\omega \cdot t) \rightarrow \text{true}$$

Compare the coefficients of cos and sin and solve for {aa, bb}

$$\text{solve}\left(aa \cdot (2 - \omega^2) + \frac{bb \cdot \omega}{4} = 2 \text{ and } bb \cdot (2 - \omega^2) - \frac{aa \cdot \omega}{4} = 0, \{aa, bb\}\right)$$

$$\rightarrow aa = \frac{-32 \cdot (\omega^2 - 2)}{16 \cdot \omega^4 - 63 \cdot \omega^2 + 64} \text{ and } bb = \frac{8 \cdot \omega}{16 \cdot \omega^4 - 63 \cdot \omega^2 + 64}$$

$$yp(t) := \frac{-32 \cdot (\omega^2 - 2)}{16 \cdot \omega^4 - 63 \cdot \omega^2 + 64} \cdot \cos(\omega \cdot t) + \frac{8 \cdot \omega}{16 \cdot \omega^4 - 63 \cdot \omega^2 + 64} \cdot \sin(\omega \cdot t) \rightarrow \text{Fertig}$$

$$y(t) := ycc(t) + yp(t) \rightarrow \text{Fertig}$$

$$16 \cdot \sqrt{127} \cdot e^{\frac{-t}{8}} \cdot \sin\left(\frac{\sqrt{127}}{8} \cdot t\right)$$

$$y(t) \rightarrow \frac{-32 \cdot (\omega^2 - 2) \cdot \cos(\omega \cdot t)}{16 \cdot \omega^4 - 63 \cdot \omega^2 + 64} + \frac{8 \cdot \omega \cdot \sin(\omega \cdot t)}{16 \cdot \omega^4 - 63 \cdot \omega^2 + 64} + \frac{16 \cdot \sqrt{127} \cdot e^{-\frac{t}{8}} \cdot \sin\left(\frac{\sqrt{127} \cdot t}{8}\right)}{127}$$

$$\text{tCollect}(yp(t)) \rightarrow \frac{-8 \cdot (4 \cdot (\omega^2 - 2) \cdot \cos(\omega \cdot t) - \omega \cdot \sin(\omega \cdot t))}{16 \cdot \omega^4 - 63 \cdot \omega^2 + 64}$$

$$f1(x) := yp(x)|_{\omega=w} \rightarrow \text{Fertig}$$

Use the following identity (DERIVE) to find the amplitude of the oscillation:

$$a \cdot \cos(x) + b \cdot \sin(x)$$

$$\sqrt{(a^2 + b^2)} \cdot \text{SIGN}(a) \cdot \cos\left(\text{ATAN}\left(\frac{b}{a}\right) - x\right)$$

$$\frac{-8 \cdot \sqrt{16 \cdot (\omega^2 - 2)^2 + \omega^2} \cdot \text{sign}(\omega^2 - 2)}{16 \cdot \omega^4 - 63 \cdot \omega^2 + 64} \rightarrow \frac{-8 \cdot \text{sign}(\omega^2 - 2)}{\sqrt{16 \cdot \omega^4 - 63 \cdot \omega^2 + 64}}$$

$$\text{zeros}\left(\frac{d}{d\omega}\left(\frac{-8 \cdot \text{sign}(\omega^2 - 2)}{\sqrt{16 \cdot \omega^4 - 63 \cdot \omega^2 + 64}}\right), \omega\right) \rightarrow \left\{-\frac{3 \cdot \sqrt{14}}{8}, 0, \frac{3 \cdot \sqrt{14}}{8}\right\}$$

$$\frac{-8 \cdot \text{sign}(\omega^2 - 2)}{\sqrt{16 \cdot \omega^4 - 63 \cdot \omega^2 + 64}} \Big|_{\omega = \frac{3 \cdot \sqrt{14}}{8}} = \frac{64 \cdot \sqrt{127}}{127}$$

$$f1(x) := yp(x)|_{\omega=w} \rightarrow \text{Fertig}$$

Use the following identity (DERIVE) to find the amplitude of the oscillation:

$$a \cdot \cos(x) + b \cdot \sin(x)$$

$$\sqrt{(a^2 + b^2)} \cdot \text{SIGN}(a) \cdot \cos\left(\text{ATAN}\left(\frac{b}{a}\right) - x\right)$$

$$\frac{-8 \cdot \sqrt{16 \cdot (\omega^2 - 2)^2 + \omega^2} \cdot \text{sign}(\omega^2 - 2)}{16 \cdot \omega^4 - 63 \cdot \omega^2 + 64} \rightarrow \frac{-8 \cdot \text{sign}(\omega^2 - 2)}{\sqrt{16 \cdot \omega^4 - 63 \cdot \omega^2 + 64}}$$

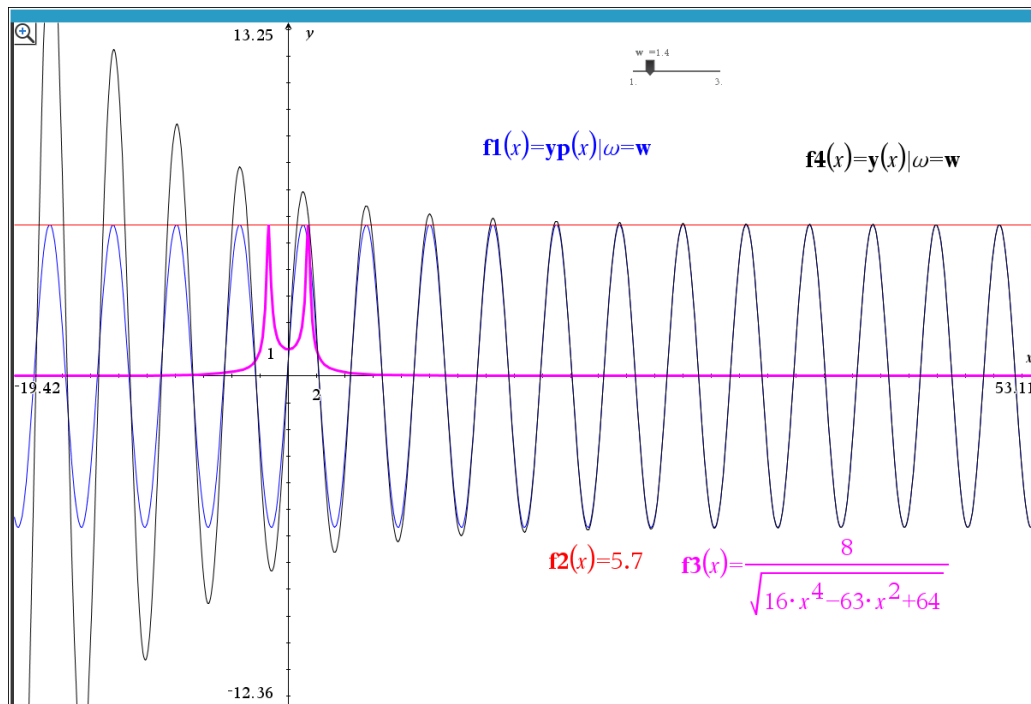
$$\text{zeros}\left(\frac{d}{d\omega}\left(\frac{-8 \cdot \text{sign}(\omega^2 - 2)}{\sqrt{16 \cdot \omega^4 - 63 \cdot \omega^2 + 64}}\right), \omega\right) \rightarrow \left\{-\frac{3 \cdot \sqrt{14}}{8}, 0, \frac{3 \cdot \sqrt{14}}{8}\right\}$$

$$\frac{-8 \cdot \text{sign}(\omega^2 - 2)}{\sqrt{16 \cdot \omega^4 - 63 \cdot \omega^2 + 64}} \Big|_{\omega = \frac{3 \cdot \sqrt{14}}{8}} = \frac{64 \cdot \sqrt{127}}{127}$$

$$f3(x) := \frac{8}{\sqrt{16 \cdot x^4 - 63 \cdot x^2 + 64}} \rightarrow \text{Fertig}$$

$$f4(x) := y(x)|_{\omega=w} \rightarrow \text{Fertig}$$

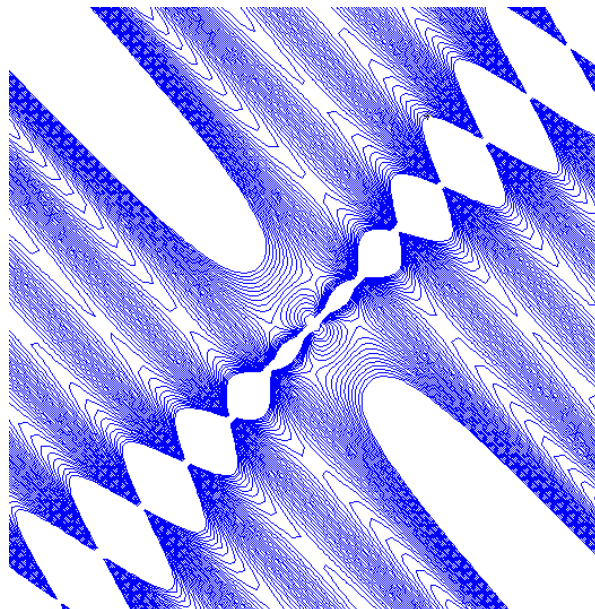
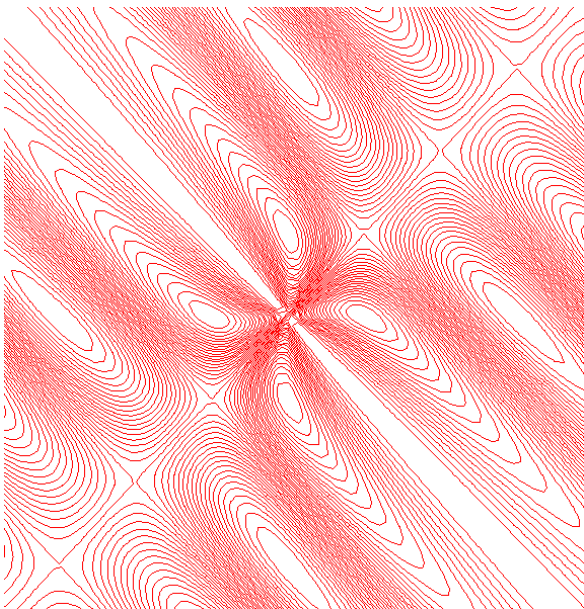
I insert a slider for ω . You can see the position for $\omega \approx 1.4$.



Pretty graphs (“Butterfly Fractals”, found anywhere in the Internet)

$$\text{VECTOR} \left(\frac{\begin{matrix} x^2 & - & y^2 \\ (x^2 - y^2) \cdot 2 \cdot \text{SIN}(x + y) \end{matrix}}{\begin{matrix} x^2 & + & y^2 \end{matrix}} = c, c, -2, 2, 0.1 \right)$$

$$\text{VECTOR} \left(\frac{\begin{matrix} x^2 & + & y^2 \\ (x^2 + y^2) \cdot \text{SIN}(x + y) \end{matrix}}{\begin{matrix} x^2 & - & y^2 \end{matrix}} = c, c, -2, 2, 0.1 \right)$$



Try this:

$$\text{VECTOR} \left(\frac{\begin{matrix} x^2 & + & x \cdot y & + & y^2 \\ (x^2 + x \cdot y + y^2) \cdot \text{SIN}(x + y) \end{matrix}}{\begin{matrix} x^2 & - & x \cdot y & + & y^2 \end{matrix}} = c, c, -2, 2, 0.1 \right)$$

Mail from our Australian friend David Halprin

Colleagues and Friends

I have spent the last 12 months delving into the inconsistencies being published and/or announced re Gravity Waves, Dark Matter, Dark Energy, Black Holes etc. with much consternation, since there seems to have been a lot of bad science fiction included as well as some fraudulent claims.

So, this third draft is worth bringing to the public's attention, in the hope that more time and money won't be wasted in some of those contentious areas of alleged research.

I have been publishing a few papers over the last year on Academia.edu so the link is included as well.

I hope that you will take the time to read right thru my paper and proffer me constructive feedback

Sincerely

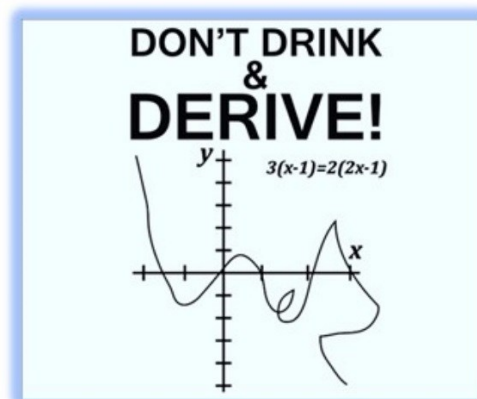
David Halprin

<https://independent.academia.edu/DavidHalprin>

Mail from Albert Rich

Josef, that's amazing! Despite all the competition, people still find *Derive* a useful mathematical assistant. But there are so many ways it could and should be improved...

Please remind all new *Derivers*:



Aloha from Hawaii,
Albert



TWO ALTERNATE APPROACHES TO CONICS

Dr. Robert Haas, USA

The essence of mathematics is its freedom.

--G. Cantor {Stewart, 62}

Traditional approaches

Traditional constructions of conics, dating back at least to Pappus of Alexandria (ca. 290 – ca. 350) involve distances measured from fixed focal points and directrix lines [Kendig, 364 – 366]. A parabola, for instance, consists of points equidistant from a fixed point and line (Figure 1a); a “stakes-and-string” ellipse consists of the points whose distances from the two focal points have a constant sum (Figure 1b).

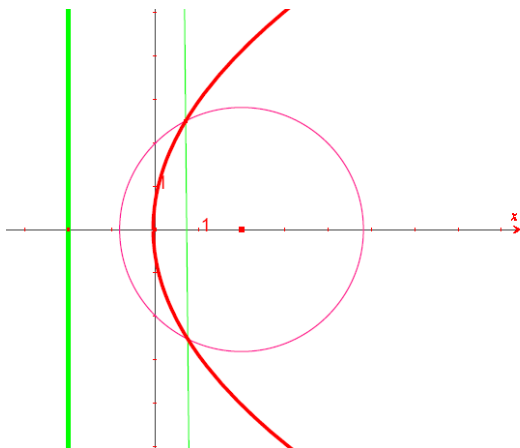


Fig. 1a. Points equidistant from a point and a circle

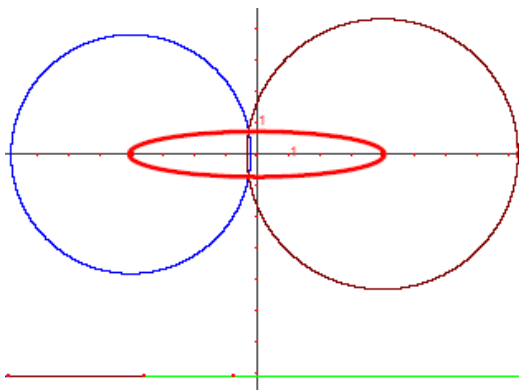


Fig. 1b. Points whose distances from the foci sum to a constant, forming an ellipse

But must one use only points and lines for reference? The present note describes two alternate approaches that replace one or both of these elements with circles. (A line is a circle of infinite radius, and a point is a circle of zero radius.)

1A. Points equidistant from a point and a circle

Replacing, first, the directrix line with a circle, consider the locus of points equidistant from a point and a circle. When the point lies outside the circle this yields one branch of a hyperbola (Figure 2a); when the point lies inside the result is an ellipse (Figure 2b). Figure 2 illustrates points equidistant from a point and a circle.

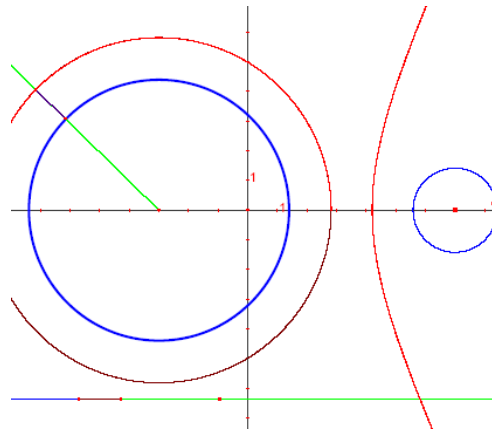


Fig. 2a: A branch of a hyperbola (point outside the circle)

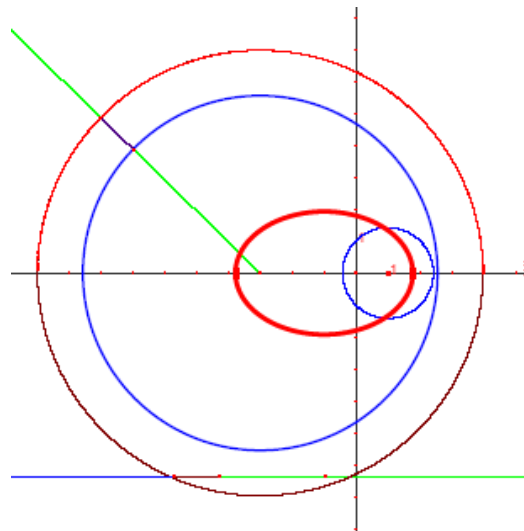


Fig. 2b: An ellipse (point inside the circle)

The analytic proof of the construction depends on two tricks, one geometric, the other algebraic. Place the circle (of radius r) with center at the origin, and the point at $(a,0)$ on the positive x axis, supposing first that $a > r$, and consider a point (x,y) equidistant from the two (Figure 3). The distance D_1 of (x,y) to the point $(a,0)$ is just the Euclidean one:

$$D_1 = \sqrt{(x-a)^2 + y^2}.$$

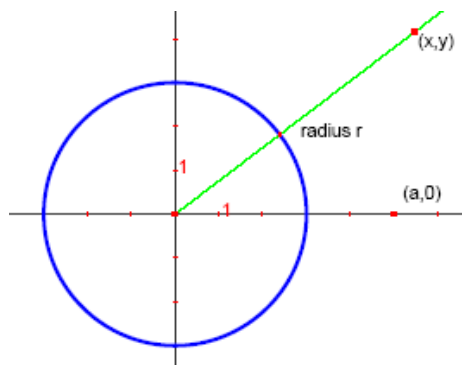


Fig. 3. Construction: the point (x,y) is equidistant from the point $(a,0)$ and the circle of radius r centered at the origin.

Now calculating the distance from (x,y) to a curved figure is in general complicated: one must (say by calculus) calculate the normal lines perpendicular to the curve, identify the one going through (x,y) , then calculate the distance along that line from the point to the curve. But the curve in question here is a circle, which has the wonderful geometric property that any line going through its center is normal, i.e. perpendicular to the circumference. So the appropriate normal here is just the line from (x,y) to $(0,0)$. And furthermore, since the distance from the circle's center to any point on its circumference is the constant radius r , the distance D_2 from (x,y) to the circle is just the Euclidean distance from (x,y) to $(0,0)$ minus r :

$$D_2 = \sqrt{(x+y)^2} - r.$$

For the construction, one now equates D_1 and D_2 , then simplifies by squaring both sides:

$$(x-a)^2 + y^2 = x^2 + y^2 + r^2 - 2r\sqrt{x^2 + y^2}$$

And here the second, algebraic, trick happens that makes the whole construction work: the x^2 and y^2 terms on the left and right sides of the equation cancel out. If they didn't, the next steps would just yield a fourth-degree mess. But since they do cancel, one can isolate the remaining radical term $-2r\sqrt{x^2 + y^2}$ on one side of the equation, square both sides again to eliminate it, and simplify the resulting second-degree equation to the final form:

$$\frac{\left(x - \frac{a}{2}\right)^2}{\left(\frac{r}{2}\right)^2} - \frac{y^2}{\frac{a^2 - r^2}{4}} = 1 \quad \text{--one branch of a hyperbola.}$$

The analysis when the point $(a,0)$ lies inside the circle--that is, $a < r$ --is exactly parallel, and yields the final form:

$$\frac{\left(x - \frac{a}{2}\right)^2}{\left(\frac{r}{2}\right)^2} + \frac{y^2}{\frac{r^2 - a^2}{4}} = 1 \quad \text{--an ellipse.}$$

When, finally, the point $(a,0)$ lies on the circle--that is, $a = r$ --the $D_1 = D_2$ equation yields $x = \sqrt{x^2 + y^2}$, forcing $y = 0$ and $x \geq 0$, the trivial special case of a ray.

1B. Points equidistant from a line and a circle

The classical construction of a parabola (Figure 1a) uses the locus equidistant from the focus point and the directrix line. Alternatively, though, enlarging the point to a circle gives a parabola, too (Figure 4).

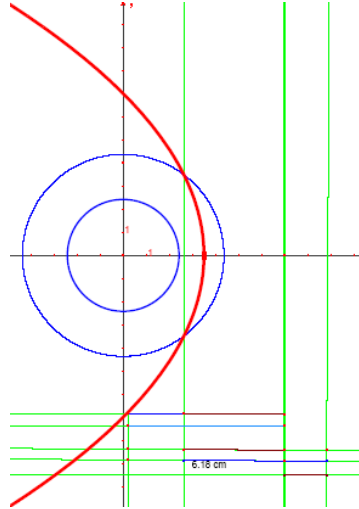


Fig. 4: Points equidistant from a circle and a line form a parabola

Algebraically, one again sets notation: circle of radius r centered at the origin, line $x = a$, for simplicity first assuming $r < a$. Then a point (x,y) of the figure lies distance $D_1 = a - x$ from the line and $D_2 = \sqrt{x^2 + y^2} - r$ from the circle. Equating $D_1 = D_2$, isolating the radical, squaring, and simplifying yields:

$$x = -\frac{y^2}{2(a+r)} + \frac{a+r}{2}$$

--- a parabola.

2A. Points equidistant from two circles

As a second alternate approach, one may replace both traditional fixed elements in the conic construction with circles. Figure 5 shows Points equidistant from two circles. The set of points equidistant from two circles forms either a branch of a hyperbola, or an ellipse.

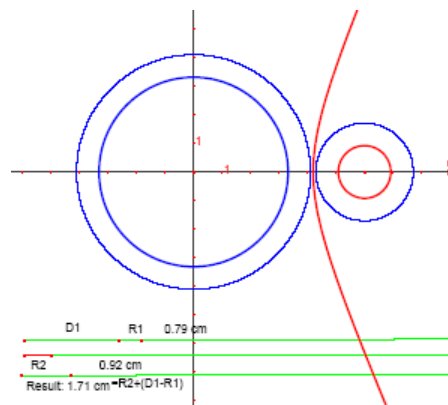


Fig. 5a: A branch of a hyperbola

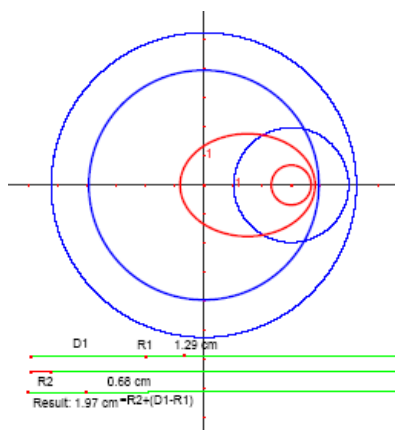


Fig. 5b: An ellipse

Algebraically, let the circles respectively have radius r (center: $(0,0)$) and s (center: $(a,0)$).

Then the distance from point (x,y) to the first circle is $D_1 = \sqrt{x^2 + y^2} - r$, to the second is

$D_2 = \sqrt{(x-a)^2 + y^2} - s$, and the equality of distances, $D_1 = D_2$, is

$$\sqrt{x^2 + y^2} - b = \sqrt{(x-a)^2 + y^2} \quad \text{where } b = s - r$$

Squaring both sides yields

$$2b\sqrt{x^2 + y^2} = a^2 - b^2 - 2ax$$

and squaring again gives

$$\frac{\left(x - \frac{a}{2}\right)^2}{\left(\frac{b}{2}\right)^2} + \frac{y^2}{b^2 - a^2} = 1,$$

which is an ellipse if $b > a$ (that is, $s > a + r$), and a branch of a hyperbola if $b < a$ (that is, $s < a + r$).

2B. Stakes-and-string variants

The traditional stakes-and-string construction gives an ellipse (Figure 1b) or hyperbola as the set of points having a fixed sum or difference of distances to two focal points. One may similarly ask what figure has a fixed sum or difference of distances to two circles. But in the notation of the previous paragraph, this simply asks $D_1 \pm D_2 = k$, and when one just incorporates k appropriately into the constant b the analysis is identical after the first squaring step.*

2C. Higher dimensions

The symmetry of the above construction now makes it easy to generalize, to find the set of points equidistant from two spheres or hyperspheres. Just as before, one places the first n -sphere (radius r) at the

* Note that distances are measured, as before, normal to the circles. An actual string wrapped around two stakes “fattened up” into cylinders of radius r and s would, in contrast, meet them tangentially, giving a much harder and messier problem.

origin, the second (radius s) on the x_1 -axis at $(a,0,0,\dots)$, and solving for points (x_j) satisfying $D_1 = D_2$ finds:

$$\frac{\left(x_1 - \frac{a}{2}\right)^2}{\left(\frac{b}{2}\right)^2} + \sum_{i>1} x_i^2 \cdot \frac{1}{\left(\frac{b^2 - a^2}{4}\right)} = 1$$

an ellipsoid if $b > a$ (that is, $s > a + r$), and a branch of a hyperboloid if $b < a$ (that is, $s < a + r$).

The construction in section 1b generalizes similarly, to show that the points (x_j) equidistant from the sphere and the hyperplane $x_1 = a$ form the circular paraboloid

$$x_1 = \sum_{i>1} x_i^2 \cdot \frac{1}{-2a - 2r} + \frac{a+r}{2}.$$

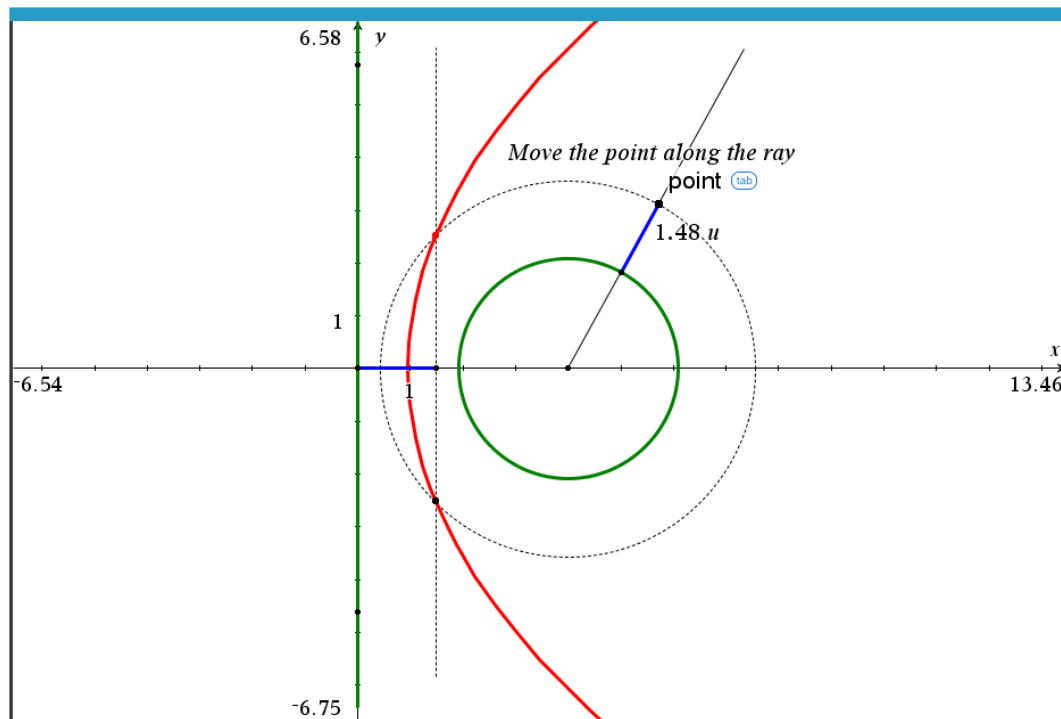
References:

Keith Kendig, Conics, Washington, D.C.: Mathematical Association of America (Dolciani Mathematical Expositions #29), 2005.

B. M. Stewart, Theory of Numbers (2nd ed.), New York: Macmillan, 1964.

Josef's remarks (sent to Robert)

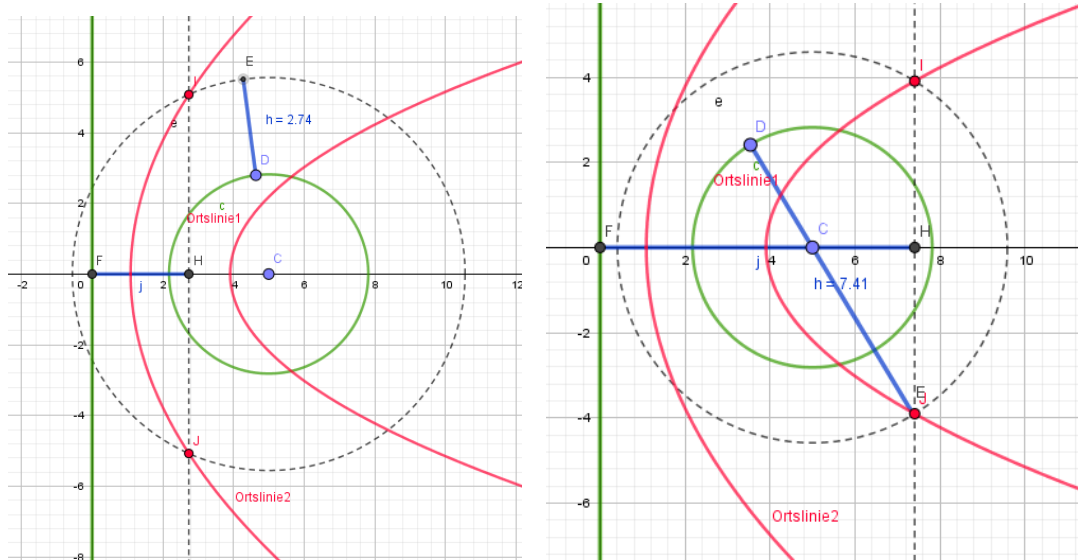
This was an interesting paper for me and I tried reproducing Robert's dynamic geometry plots using tools which I have available:



This done with TI-Nspire (line and circle Robert's 1B))

Then I worked with GeoGebra (which is free for everybody).

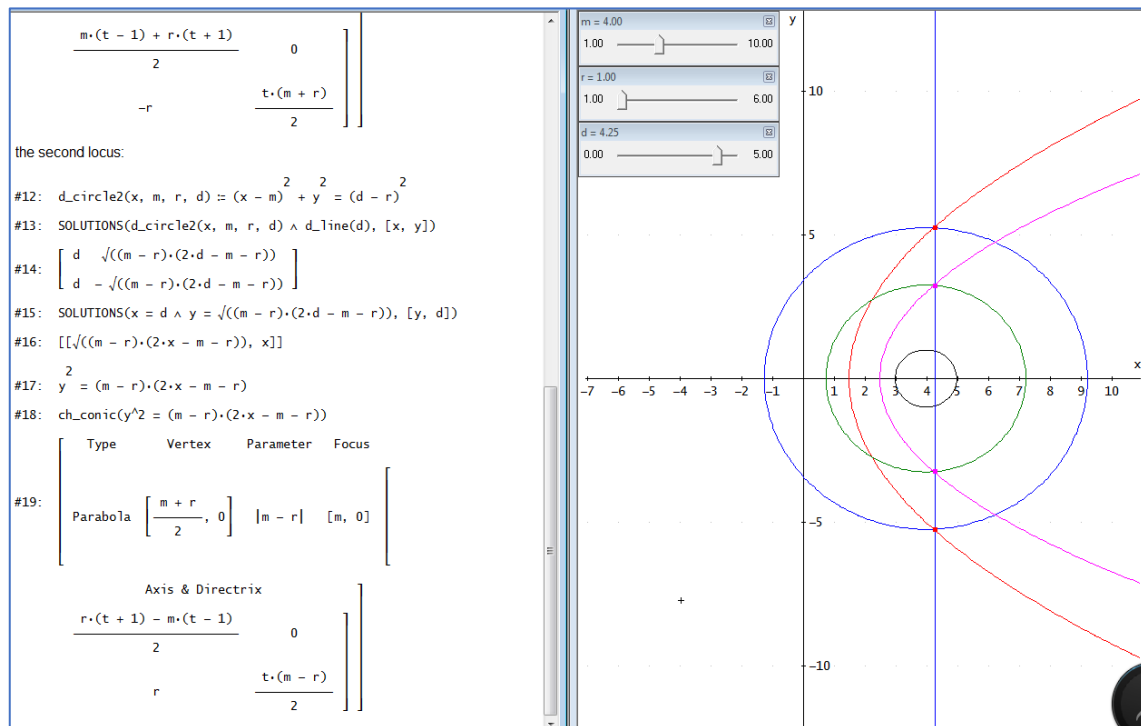
Big surprise! The locus tool presented two loci (red curves). The left one was the parabola, which I had expected, but what about the right one?



You can see, when moving point E in the opposite direction, the dashed circle is “inverted” and the intersection points form the second parabola. Interesting, isn’t it?

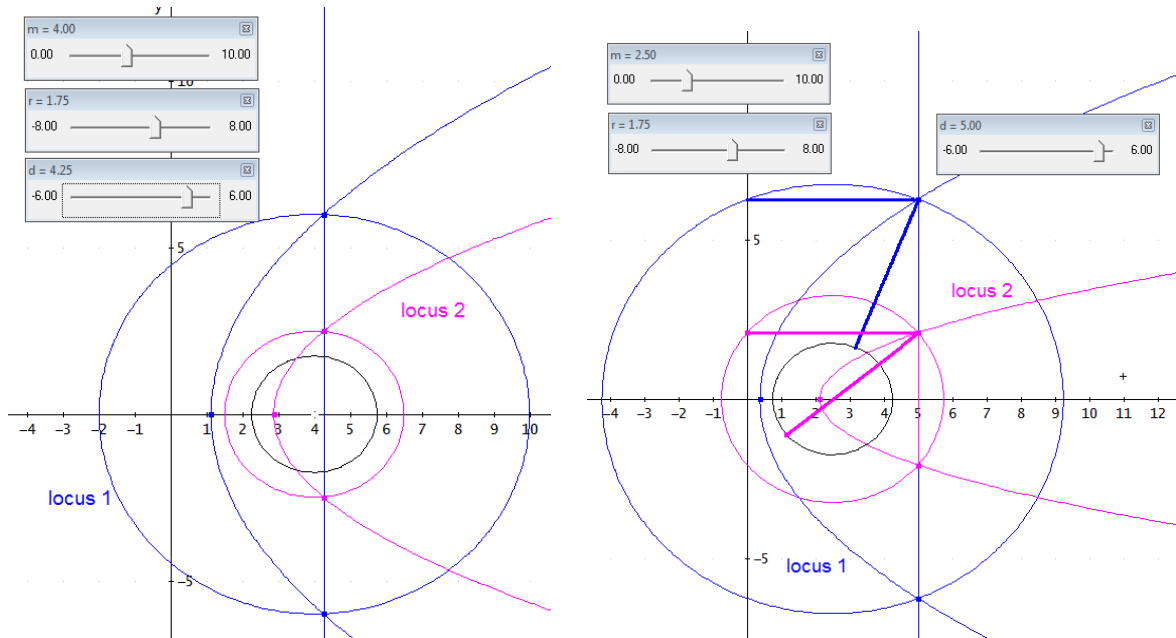
I admit that I didn’t try to find its equation. Next time?

And I worked with my favorite program DERIVE. Here I can easily combine CAS-calculation and simulating Dynamic Geometry simultaneously using sliders for the parameters.



A copy of the screen is above: left hand side shows the calculation which can be accompanied simultaneously by the plot. I could add the segments to underline the equal distances to circle and line. The last lines apply a self-made tool for identifying all kinds of conics. This is again Robert's 1B paragraph.

Calculation and plot of the second locus (magenta) are presented. The necessary requirements are fulfilled – same distance from line and circle (applied in the other direction!)



Conics1.dfw

DERIVE calculation accompanied by the respective plots applying sliders for all parameters follows:

$$\#1: c(m, r) := (x - m)^2 + y^2 = r^2$$

base circles (black)

$$\#2: [bc1 := c(0, r), bc2 := c(a, s)]$$

distant circles (blue)

$$\#3: [dc1 := c(0, r + d), dc2 := c(a, s + d), dc11 := c(0, r - d), dc22 := c(a, s - d)]$$

$$\#4: pts1 := SOLUTIONS(dc1 \wedge dc2, [x, y])$$

$$\#5: locus1 := SOLUTIONS(dc1 \wedge dc2, [y, d])$$

$$\#6: locus1$$

$$\#7: \left[\begin{array}{l} \frac{\sqrt{((a - r)^2 + 2 \cdot r \cdot s - s^2) \cdot (4 \cdot x^2 - 4 \cdot a \cdot x + a^2 - r^2 + 2 \cdot r \cdot s - s^2)}}{2 \cdot |r - s|} \quad \frac{2 \cdot a \cdot x - a^2 - r^2 + s^2}{2 \cdot (r - s)} \\ - \frac{\sqrt{((a - r)^2 + 2 \cdot r \cdot s - s^2) \cdot (4 \cdot x^2 - 4 \cdot a \cdot x + a^2 - r^2 + 2 \cdot r \cdot s - s^2)}}{2 \cdot |r - s|} \quad \frac{2 \cdot a \cdot x - a^2 - r^2 + s^2}{2 \cdot (r - s)} \end{array} \right]$$

$$\#8: (y = locus1_{1,1})^2$$

$$\#8: (y = \text{locus1})_{1,1}^2$$

LOCUS 1 (red)

$$\#9: y = \frac{(a^2 - r^2 + 2 \cdot r \cdot s - s^2) \cdot (4 \cdot x^2 - 4 \cdot a \cdot x + a^2 - r^2 + 2 \cdot r \cdot s - s^2)}{4 \cdot (r - s)^2}$$

$$\#10: \text{pts2} := \text{pts1} := \text{SOLUTIONS}(\text{dc1} \wedge \text{dc22}, [x, y])$$

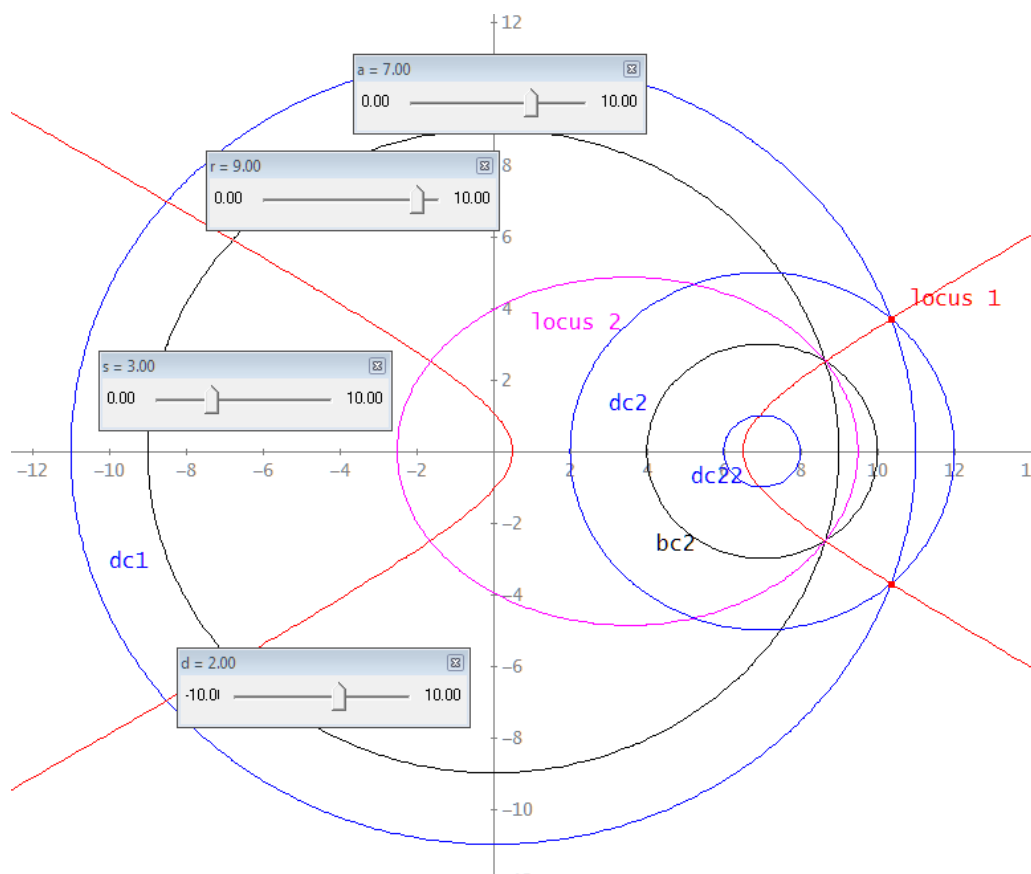
$$\#11: \text{locus2} := \text{SOLUTIONS}(\text{dc1} \wedge \text{dc22}, [y, d])$$

$$\#12: (y = \text{locus2})_{1,1}^2$$

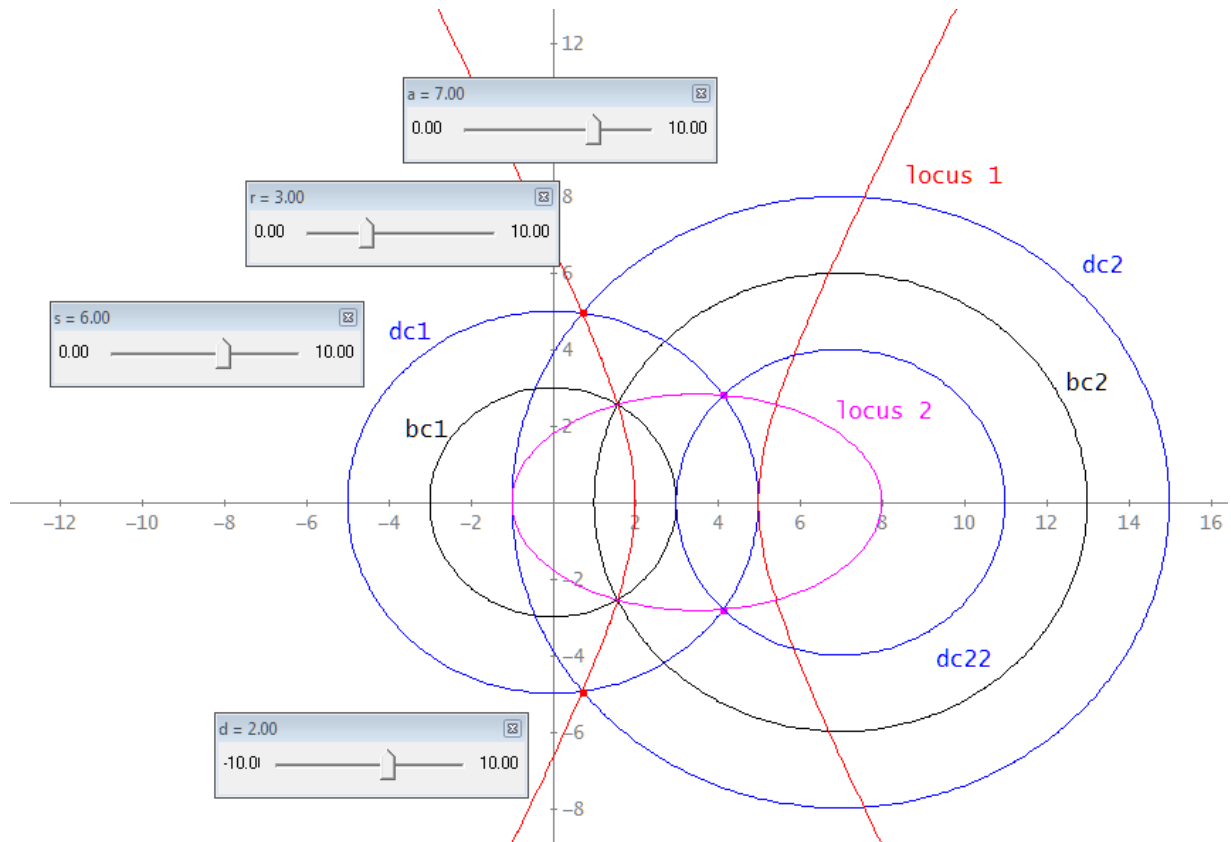
LOCUS 2 (magenta)

$$\#13: y = \frac{(a^2 - r^2 - 2 \cdot r \cdot s - s^2) \cdot (4 \cdot x^2 - 4 \cdot a \cdot x + a^2 - r^2 - 2 \cdot r \cdot s - s^2)}{4 \cdot (r + s)^2}$$

This is an interesting plot, because the result of the calculation of the second locus (the ellipse) shows no intersection points of the two generating circles – complex points? See the next plot!



Now we have real intersection points generating the ellipse:



$$\#14: \text{SOLUTIONS}(x^2 + y^2 = (9 + d)^2 \wedge (x - 7)^2 + y^2 = (3 - d)^2, [y, d])$$

$$\#15: \begin{bmatrix} \frac{\sqrt{95} \cdot \sqrt{-4x^2 + 28x + 95}}{24} & \frac{14x - 121}{24} \\ -\frac{\sqrt{95} \cdot \sqrt{-4x^2 + 28x + 95}}{24} & \frac{14x - 121}{24} \end{bmatrix}$$

$$\#16: \left(y = \frac{\sqrt{95} \cdot \sqrt{-4x^2 + 28x + 95}}{24} \right)^2$$

$$\#17: y^2 = -\frac{95 \cdot (4x^2 - 28x - 95)}{576}$$

the intersection points will appear on the ellipse for $d < 0$!!

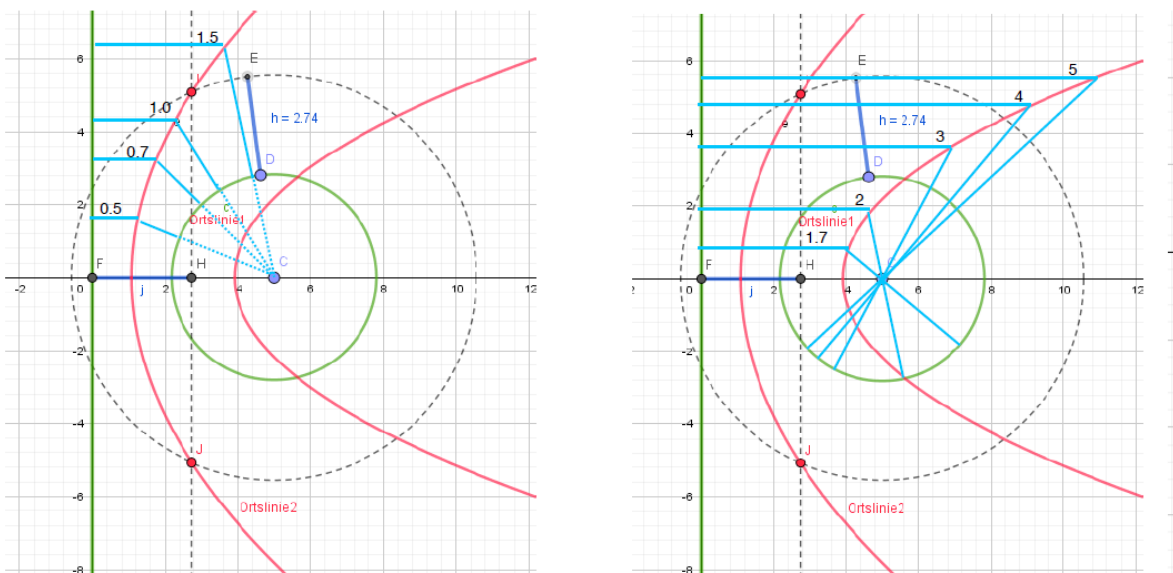
Mail from Robert

Thank you for your February 2 email and its attached conics.pdf--how kind of you to revive my conics work from the past, and then extend it with your more modern computer tools!

Two parabolas

The two parabolas from the GeoGebra plot on p. 23 can be explained readily enough: The left-hand parabola is the correct one: I've marked in the left-hand figure below how four points on the curve satisfy the defining condition, that the distance measured left to the y-axis is the same as the distance to the circle (measured along the line that is perpendicular to the circle because it goes through its center), for distances 0.5, 0.7, 1.0, and 1.5 cm. The right-hand figure below shows how the program has *mistakenly* constructed the right-hand parabola, to satisfy the condition that the distance measured left to the y-axis is the same as the distance to the *remotest point* of the circle (again measured along the line going through its center), for distances 1.7, 2, 3, 4, and 5 cm.

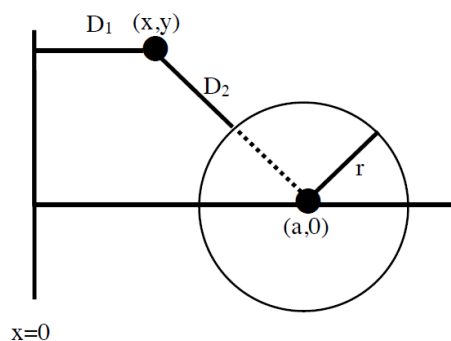
GeoGebra Parabolas



To our human common sense, this second parabola is a mistake as absurd as if one would ask how far Cleveland is above sea level, and rather than saying a couple hundred meters—the height that the water flowing out of Lake Erie drops down Niagara Falls on its way to the Atlantic Ocean--would instead answer that it's almost 13,000 km, the distance measured through the center of the earth to sea level at the antipodal point in the Indian Ocean southwest of Australia! Only a (badly programmed) computer would make this mistake! There *is* a point on the GeoGebra circle at the specified distance--but it's the *maximal* distance over this set of points, not the minimal one that is understood implicitly in asking for the distance of the whole set. There are probably also points on the circle at *any other distance between* those of the two parabolas. The GeoGebra programmers have simply not thought through the situation, and tested their software well enough, to catch this problem, so I think you are doing your readers a genuine service by pointing it out.

Analytical solutions

The analytical solutions to the GeoGebra parabolas are as follows:



$$D_1 = x$$

$$D_2 = \sqrt{[(x-a)^2 + y^2]} - r$$

(minimum distance to circle)

Equating D_1 and D_2

$$x + r = \sqrt{[(x-a)^2 + y^2]}$$

Squaring:

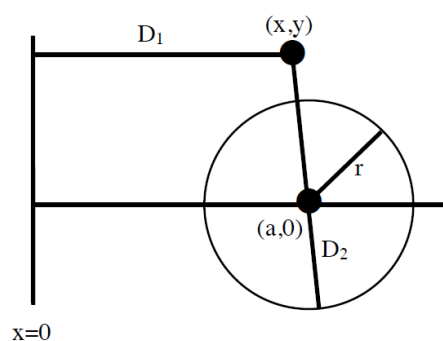
$$x^2 + 2rx + r^2 = x^2 - 2ax + a^2 + y^2$$

Simplifying:

$$x(2r+2a) = a^2 + y^2 - r^2$$

Solution:

$$x = \frac{y^2}{2(a+r)} + \frac{a-r}{2}$$



$$D_1 = x$$

$$D_2 = \sqrt{[(x-a)^2 + y^2]} + r$$

(maximum distance to circle)

Equating D_1 and D_2

$$x - r = \sqrt{[(x-a)^2 + y^2]}$$

Squaring:

$$x^2 - 2rx + r^2 = x^2 - 2ax + a^2 + y^2$$

Simplifying:

$$x(-2r+2a) = a^2 + y^2 - r^2$$

Solution:

$$x = \frac{y^2}{2(a-r)} + \frac{a+r}{2}$$

A question for the future

As I explain on pp. 18-19, my analysis depends crucially on a special property of the circle: that every line through its center is normal to the circle (and hence that the distance from any outside point to a circle is simply the distance to its center, minus the radius). So, there is no way, analytically, to attack a simple generalization of my questions like: What is the locus of points equidistant from a line and an *ellipse*? But your computer has no such limitations! I think, by continuity, the resulting locus will be trapped between the loci generated by the line and either the inscribed or the circumscribed circle of the ellipse. It might be fun to explore how weird a result you can obtain.

Do you know the online *Journal of Humanistic Mathematics*? The latest issue that came out last week has two of my articles: *JHM* 12(1) (2022) 5-23 describes my mathematical theory of painting, poetry, and music, while *JHM* 12(1) (2022) 217-218 is a note on the geometry of social distancing. Page 218 lists my articles from over the years in this friendly journal: The Raphael article (which I sent you once) considers on its pp. 14-15 how by projective geometry to undo the distortion in a plane figure when viewed from an angle, while my cello quartics article addresses how to find the line tangent to a curve at *two* points. Can modern computer geometry programs handle questions like these?

Dear Robert,

many thanks for your answer including your comments (and pointing to my typos – shame on me!!). I understand that your interpretation of distance is the minimum distance which is the perpendicular distance.

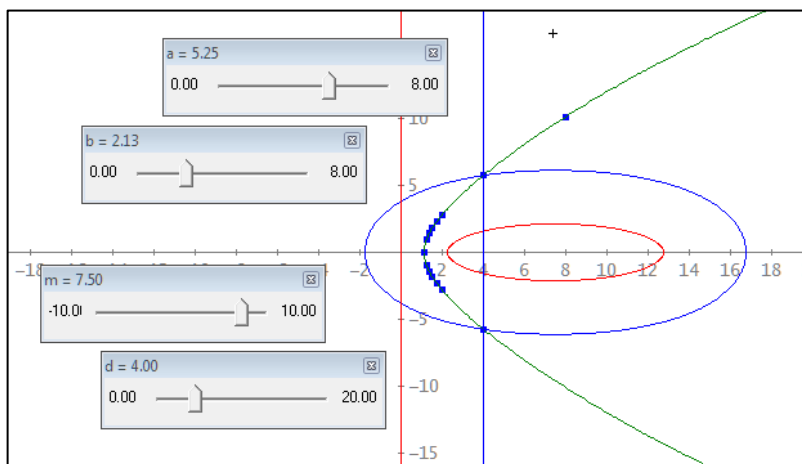
I like how you distinguish between minimum and maximum distance. I believe that the software behind GeoGebra interprets the circle consisting of two half circles?? Because it is not a function but a relation. So, it does not distinguish between the semicircles – or similar. It is an interesting problem and if you don't mind, I would add your very valuable comments to the article. Maybe that we will get some reactions.

Robert's "Questions for the Future"

There are two challenges in Robert's mail:

- What is the locus of points equidistant from a line and an *ellipse*?
- How to find the line tangent to a curve at *two* points?

I'd like to begin dealing with the first one.



We need the parallel curve of the function $[x(t), y(t)]$ given in parameter form:

$$x_d(t) = x(t) - \frac{d \cdot y'(t)}{\sqrt{x'(t)^2 + y'(t)^2}}$$

$$y_d(t) = y(t) + \frac{d \cdot x'(t)}{\sqrt{x'(t)^2 + y'(t)^2}}$$

I did it as I described in my lecture "Narrowing the Gap between Computer Algebra and Dynamic Geometry" given at DES-TIME-2006, Dresden. I split the screen. In the Algebra Window I performed the calculations with general parameters and in the 2D-Plot Window I plotted simultaneously the progress of the calculations with sliders introduced for the parameters.

```
[e1] := [m + a·COS(t), b·SIN(t)], l := [d, s]
```

$$e1p := \left[m + a \cdot \cos(t) + \frac{d \cdot b \cdot \cos(t)}{\sqrt{(a \cdot \sin(t))^2 + (b \cdot \cos(t))^2}}, b \cdot \sin(t) + \frac{d \cdot a \cdot \sin(t)}{\sqrt{(a \cdot \sin(t))^2 + (b \cdot \cos(t))^2}} \right]$$

```
[hCross, vCross]
```

```
[1.25, 1]
```

```
[ 1.25  1 ]
```

```
[ 1.25 -1 ]
```

```
[1.375, 1.423076923]
```

I plot $e1$, $y = 0$ (red) and l (blue) and move the slider for d (with any choice for the ellipse axes). I set the cross on the intersection points and approximate $[hcross, vcross]$ in the Algebra Window, author the respective pair of points which can immediately be plotted. In this manner I can plot a set of points which will lie on the locus to be calculated (if possible?)

Now, I try to find the locus, equating the first and second coordinates of the two blue curves and solving the system of with three unknowns (d, s, t) trying to eliminate d in order to end with a parameter form of the desired locus.

The expressions appearing are pretty huge, so I don't show them as a whole.

$$\left[\text{linep}_1 = \text{ellp}_1, \text{linep}_2 = \text{ellp}_2 \right]$$

$$\text{SOLVE}(\text{linep}_1 = \text{ellp}_1 \wedge \text{linep}_2 = \text{ellp}_2, [s, d])$$

$$s = \frac{b \cdot \sqrt{(b \cdot \cos(t)^2 + a \cdot \sin(t)^2)} \cdot (m \cdot \cos(t) + a) + a \cdot b \cdot \cos(t)^3 + b^2 \cdot \sin(t)^2}{a \cdot \sin(t) \cdot \sqrt{(b \cdot \cos(t)^2 + a \cdot \sin(t)^2)}}$$

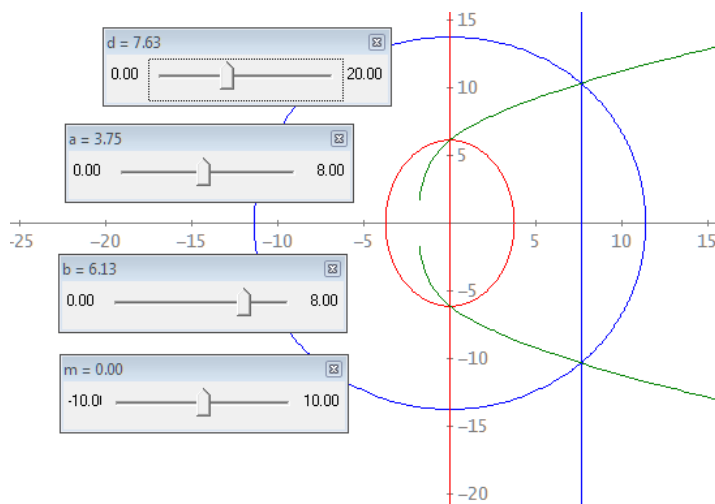
The system can be solved for s and d . I substitute in $\text{linep}[d, s]$ and receive a bulky expression with trig functions. To make the expression smaller I substitute $\text{ACOS}(t)$ for t . I was very happy that the graph (green) passed the points previously plotted – but only in the upper half plane. No problem to add the lower part of the locus, too.

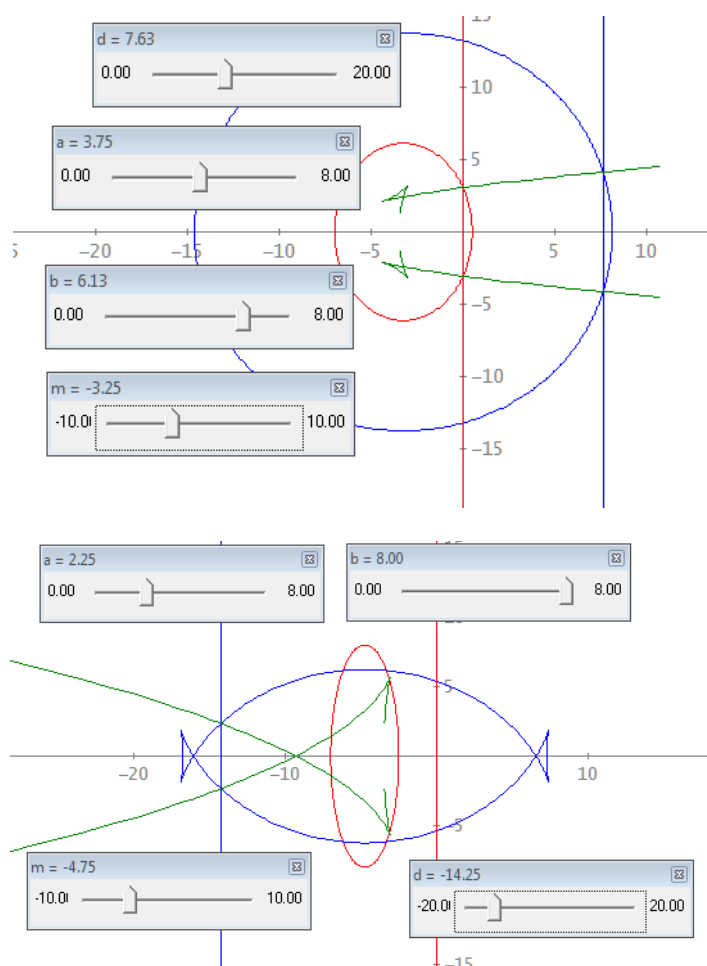
$$\left[\sqrt{(b \cdot \cos(t)^2 + a \cdot \sin(t)^2)} \cdot \left(\frac{b \cdot \cot(t)^2}{a} + \frac{b \cdot m \cdot \cos(t)}{a \cdot \sin(t)^2} \right) + \frac{b^2 \cdot \cos(t)^3}{a \cdot \sin(t)^2} + \frac{b^2 \cdot m \cdot \cot(t)^2}{a} + \sqrt{(b \cdot \cos(\text{ACOS}(t))^2 + a \cdot \sin(\text{ACOS}(t))^2)} \cdot \left(\frac{b \cdot \cot(\text{ACOS}(t))^2}{a} + \frac{b \cdot m \cdot \cos(\text{ACOS}(t))}{a \cdot \sin(\text{ACOS}(t))^2} \right) + \frac{b^2 \cdot \cos(\text{ACOS}(t))}{a \cdot \sin(\text{ACOS}(t))} \right]$$

This is the parameter form of the locus with parameter t :

$$\left[\frac{(a \cdot t + m) \cdot (b \cdot t \cdot \sqrt{(a \cdot (1 - t)^2 + b \cdot t^2)} + a \cdot (1 - t)^2 + b \cdot t^2)}{a \cdot (t + 1) \cdot (1 - t)}, \frac{b \cdot (a + m \cdot t) \cdot \sqrt{(a \cdot (1 - t)^2 + b \cdot t^2)}}{a \cdot (t + 1) \cdot (1 - t)} \right]$$

Changing the parameter leads to unexpected forms of the locus:





I was not able to find the locus in case of two ellipses and their respective parallel curves. All dynamic geometry programs refused to fix the intersection point of the two parallel curves when changing the distance d and consequently, they did not plot the locus. Maybe that one of our readers has a better idea how to tackle this “challenge”.

Robert’s second question concerns his “Cello Quartics Article”.

You can find his paper at

<https://scholarship.claremont.edu/jhm/vol8/iss1/15/>

I will come back to this article published in *Journal of Humanistic Mathematics* in the next DNL.



More links to the “scholarship.claremont”-websites are given on the first inside page.

Working through Bhuvanesh Bhatt's Mathtool-library for TI-92 I came across a short chapter "Optimization", containing functions for Lagrange Multipliers, a special minimizing a function method and the Simplex Method. I remembered that we had these topics in earlier DNLs (see the references) but nevertheless I had the idea that an update might be useful. Josef

Optimization Problems (Lagrange Multipliers, Simplex Method, ...)

Bhuvanesh Bhatt & Josef Böhm

From Bhuvanesh Bhatt's Mathtool Library:

LagrMult($f(x,y,...)$, { $\text{cond1}, \text{cond2}, \dots$ }, { x, \dots }) finds extrema for $f(x,y,...)$ subject to the conditions { $\text{cond1}=0, \text{cond2}=0, \dots$ }

Needs: list2eqn

Example: $\text{LagrMult}(x^2+y^2, \{x^2+y-1\}, \{x,y\}) \Rightarrow x = \sqrt{2}/2$ and $y = 1/2$ and ...

Note: If you want **LagrMult** to return the maximum/minimum values of $f(x,y,...)$ in addition to the values of x,y,\dots , open the function in the Program Editor and delete the "Return hh" line. The extrema returned may be local or global extrema. You can use the Hessian to determine the type of extremum.

```

F1 Control F2 I/O Var F3 Find... F4 Mode
:lagrmult(ff,cnd,vars)
:Func
:@Bhuvanesh Bhatt
:Local λ, hh, hhh, opt, lopt, dv, i, j, header:
dim(vars)→dv:seq(expr("λ"&string(i)),i,
1,dim(cnd))→λ:ff+sum(λ*cnd)→hh:augment
(vars,λ)→vars:mathtool\list2eqn(seq(d
(hh,vars[i])=0,i,1,dim(vars)))→hh:expr(
"solve("&string(hh)&","&string(vars)&")
")→hh
:Return hh
:expr("exp▶list("&string(hh)&","&string(

```

```

F1 Control F2 I/O Var F3 Find... F4 Mode
:expr("exp▶list("&string(hh)&","&string(
vars)&")")→hh
:O→lopt
:For i,1,dim(hh)[1]
: ff→opt
: For j,1,dv
: limit(opt,vars[j],hh[i,j])→opt
: EndFor
:augment(lopt,(opt))→lopt
:EndFor
:list▶mat(lopt,1)→lopt
:augment(vars,{"Opt"})→header
:list▶mat(header,dim(header))→header
:Return (augment(headerT, (augment(hh,lop
t))T))
:EndFunc

```

```

F1 Control F2 I/O Var F3 Find... F4 Mode
: ff→opt
: For j,1,dv
: limit(opt,vars[j],hh[i,j])→opt
: EndFor
:augment(lopt,(opt))→lopt
:EndFor
:list▶mat(lopt,1)→lopt
:augment(vars,{"Opt"})→header
:list▶mat(header,dim(header))→header
:Return (augment(headerT, (augment(hh,lop
t))T))
:EndFunc

```

```

F1 Algebra F2 Calc F3 Other F4 PrgmIO F5 Clean Up
■ lagrmult(x2+y2, {x2+y-1}, {x y})
x =  $\frac{-\sqrt{2}}{2}$  and y = 1/2 and λ1 = -1 or x = 0
■ lagrmult(x2+y2, {x2+y-1}, {x y})
x y λ1 "Opt"
 $\frac{\sqrt{2}}{2}$  1/2 -1 3/4
...ult(x2+y2, {x2+y-1}, {x, y})

```

I tried this function and the first part worked pretty well as the first result is showing. Then I deactivated the first "Return hh" by making it to a comment @ Return hh". And this did not work. So, I wrote another ending of the function. Here the result(s) are presented in form of a matrix showing all variables together with the function value of the function to be optimized under given restriction(s).

In DNL#112 you can find Don Phillips' respective tool for TI-Nspire. Next page shows two examples treated with his function. Unfortunately, I missed the occasion to add a function for DERIVE in DNL#112. As I don't want to repeat my omission, I will include the DERIVE procedure now. Additionally, I will provide graphic descriptions of the problems and their solutions.

$\text{lagrangemulti}(x^2+y^2, x^2+y-1=0)$	<table border="1"> <thead> <tr> <th>"Answers"</th> <th>"Value"</th> </tr> </thead> <tbody> <tr> <td>$x=\frac{\sqrt{2}}{2}$ and $y=\frac{1}{2}$ and $\lambda=1$</td> <td>$\frac{3}{4}$</td> </tr> <tr> <td>$x=0$ and $y=1$ and $\lambda=2$</td> <td>1</td> </tr> <tr> <td>$x=\frac{\sqrt{2}}{2}$ and $y=\frac{1}{2}$ and $\lambda=1$</td> <td>$\frac{3}{4}$</td> </tr> </tbody> </table>	"Answers"	"Value"	$x=\frac{\sqrt{2}}{2}$ and $y=\frac{1}{2}$ and $\lambda=1$	$\frac{3}{4}$	$x=0$ and $y=1$ and $\lambda=2$	1	$x=\frac{\sqrt{2}}{2}$ and $y=\frac{1}{2}$ and $\lambda=1$	$\frac{3}{4}$		
"Answers"	"Value"										
$x=\frac{\sqrt{2}}{2}$ and $y=\frac{1}{2}$ and $\lambda=1$	$\frac{3}{4}$										
$x=0$ and $y=1$ and $\lambda=2$	1										
$x=\frac{\sqrt{2}}{2}$ and $y=\frac{1}{2}$ and $\lambda=1$	$\frac{3}{4}$										
$\text{lagrangemulti}\left(\sqrt{x^2+y^2+z^2}, \begin{cases} x^2+y^2+z^2 \\ x+y=4 \end{cases}, z=16\right)$	<table border="1"> <thead> <tr> <th>"Answers"</th> <th>"Value"</th> </tr> </thead> <tbody> <tr> <td>$x=(\sqrt{3}-2)$ and $y=\sqrt{3}+2$ and $z=1$ and $\lambda_1=\frac{\sqrt{15}}{30}$ and $\lambda_2=0$</td> <td>$\sqrt{15}$</td> </tr> <tr> <td>$x=2$ and $y=2$ and $z=4$ and $\lambda_1=\frac{\sqrt{6}}{6}$ and $\lambda_2=\frac{-\sqrt{6}}{2}$</td> <td>$2\sqrt{6}$</td> </tr> <tr> <td>$x=\sqrt{3}+2$ and $y=(\sqrt{3}-2)$ and $z=1$ and $\lambda_1=\frac{\sqrt{15}}{30}$ and $\lambda_2=0$</td> <td>$\sqrt{15}$</td> </tr> </tbody> </table>	"Answers"	"Value"	$x=(\sqrt{3}-2)$ and $y=\sqrt{3}+2$ and $z=1$ and $\lambda_1=\frac{\sqrt{15}}{30}$ and $\lambda_2=0$	$\sqrt{15}$	$x=2$ and $y=2$ and $z=4$ and $\lambda_1=\frac{\sqrt{6}}{6}$ and $\lambda_2=\frac{-\sqrt{6}}{2}$	$2\sqrt{6}$	$x=\sqrt{3}+2$ and $y=(\sqrt{3}-2)$ and $z=1$ and $\lambda_1=\frac{\sqrt{15}}{30}$ and $\lambda_2=0$	$\sqrt{15}$		
"Answers"	"Value"										
$x=(\sqrt{3}-2)$ and $y=\sqrt{3}+2$ and $z=1$ and $\lambda_1=\frac{\sqrt{15}}{30}$ and $\lambda_2=0$	$\sqrt{15}$										
$x=2$ and $y=2$ and $z=4$ and $\lambda_1=\frac{\sqrt{6}}{6}$ and $\lambda_2=\frac{-\sqrt{6}}{2}$	$2\sqrt{6}$										
$x=\sqrt{3}+2$ and $y=(\sqrt{3}-2)$ and $z=1$ and $\lambda_1=\frac{\sqrt{15}}{30}$ and $\lambda_2=0$	$\sqrt{15}$										
$\text{lagrangemulti}(x \cdot y, 4 \cdot x^2+y^2-4=0)$	<table border="1"> <thead> <tr> <th>"Answers"</th> <th>"Value"</th> </tr> </thead> <tbody> <tr> <td>$x=\frac{-\sqrt{2}}{2}$ and $y=-\sqrt{2}$ and $\lambda=\frac{1}{4}$</td> <td>1</td> </tr> <tr> <td>$x=\frac{-\sqrt{2}}{2}$ and $y=\sqrt{2}$ and $\lambda=\frac{-1}{4}$</td> <td>-1</td> </tr> <tr> <td>$x=\frac{\sqrt{2}}{2}$ and $y=-\sqrt{2}$ and $\lambda=\frac{-1}{4}$</td> <td>-1</td> </tr> <tr> <td>$x=\frac{\sqrt{2}}{2}$ and $y=\sqrt{2}$ and $\lambda=\frac{1}{4}$</td> <td>1</td> </tr> </tbody> </table>	"Answers"	"Value"	$x=\frac{-\sqrt{2}}{2}$ and $y=-\sqrt{2}$ and $\lambda=\frac{1}{4}$	1	$x=\frac{-\sqrt{2}}{2}$ and $y=\sqrt{2}$ and $\lambda=\frac{-1}{4}$	-1	$x=\frac{\sqrt{2}}{2}$ and $y=-\sqrt{2}$ and $\lambda=\frac{-1}{4}$	-1	$x=\frac{\sqrt{2}}{2}$ and $y=\sqrt{2}$ and $\lambda=\frac{1}{4}$	1
"Answers"	"Value"										
$x=\frac{-\sqrt{2}}{2}$ and $y=-\sqrt{2}$ and $\lambda=\frac{1}{4}$	1										
$x=\frac{-\sqrt{2}}{2}$ and $y=\sqrt{2}$ and $\lambda=\frac{-1}{4}$	-1										
$x=\frac{\sqrt{2}}{2}$ and $y=-\sqrt{2}$ and $\lambda=\frac{-1}{4}$	-1										
$x=\frac{\sqrt{2}}{2}$ and $y=\sqrt{2}$ and $\lambda=\frac{1}{4}$	1										
$\text{lagrangemulti}(8 \cdot x \cdot y \cdot z, 16 \cdot x^2+4 \cdot y^2+9 \cdot z^2=144)$	<table border="1"> <thead> <tr> <th>"Answers"</th> <th>"Value"</th> </tr> </thead> <tbody> <tr> <td>$x=-3$ and $y=0$ and $z=0$ and $\lambda=0$</td> <td>0</td> </tr> <tr> <td>$x=-\sqrt{3}$ and $y=-2\sqrt{3}$ and $z=\frac{-4\sqrt{3}}{3}$ and $\lambda=\frac{-2\sqrt{3}}{3}$</td> <td>$-64\sqrt{3}$</td> </tr> </tbody> </table>	"Answers"	"Value"	$x=-3$ and $y=0$ and $z=0$ and $\lambda=0$	0	$x=-\sqrt{3}$ and $y=-2\sqrt{3}$ and $z=\frac{-4\sqrt{3}}{3}$ and $\lambda=\frac{-2\sqrt{3}}{3}$	$-64\sqrt{3}$				
"Answers"	"Value"										
$x=-3$ and $y=0$ and $z=0$ and $\lambda=0$	0										
$x=-\sqrt{3}$ and $y=-2\sqrt{3}$ and $z=\frac{-4\sqrt{3}}{3}$ and $\lambda=\frac{-2\sqrt{3}}{3}$	$-64\sqrt{3}$										

The DERIVE program:

```

lagr_multi(f, cond, vars, dc, dv, pars, vvars, ggrad, sols, vals, header) :=
  Prog
  pars := [m1, m2, m3, m4, m5]
  i := 1
  dc := DIM(cond)
  dv := DIM(vars)
  Loop
  If i > dc exit
  cond_i := LHS(cond_i) - RHS(cond_i)
  i := i + 1
  ff := f + Σ(pars_i·cond_i, i, 1, dc)
  vvars := APPEND(vars, pars_i[1, ..., dc])
  ggrad := VECTOR(∂(ff, vvars_i), i, DIM(vvars))
  sols := SOLUTIONS(ggrad, vvars)
  vals := VECTOR(SUBST(f, vars, sols_i_i[1, ..., dv]), i, DIM(sols))
  vals := APPEND(sols', [vals])
  header := APPEND(VECTOR(STRING(v), v, vvars), ["Value"])
  RETURN APPEND([header], vals)

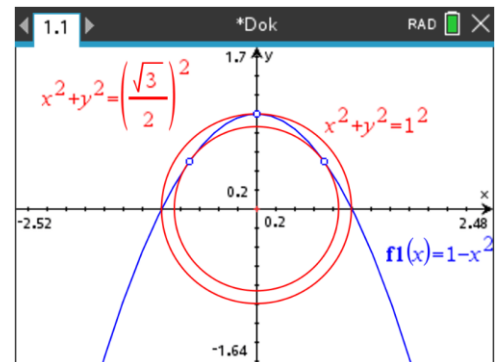
```

$$\text{lagr_multi}(x^2 + y^2, [x^2 + y - 1 = 0], [x, y]) = \begin{bmatrix} x & y & m1 & \text{Value} \\ 0 & 1 & -2 & 1 \\ \frac{\sqrt{2}}{2} & \frac{1}{2} & -1 & \frac{3}{4} \\ -\frac{\sqrt{2}}{2} & \frac{1}{2} & -1 & \frac{3}{4} \end{bmatrix}$$

The task behind the solution could read as follows:

Which are the circles with center in the origin osculating the parabola $y = x^2 + 1$ have minimum/maximum radius?

Answers: $r = 1, r = \frac{\sqrt{3}}{2}$



Which points on the intersection curve of the paraboloid and the vertical plane have minimum/maximum distance to the origin?

```
lagn_multi(sqrt(x^2 + y^2 + z^2), [x^2 + y^2 + 2*z = 16, x + y = 4], [x, y, z])
```

```
[[x, y, z, m1, m2, Value], []]
```

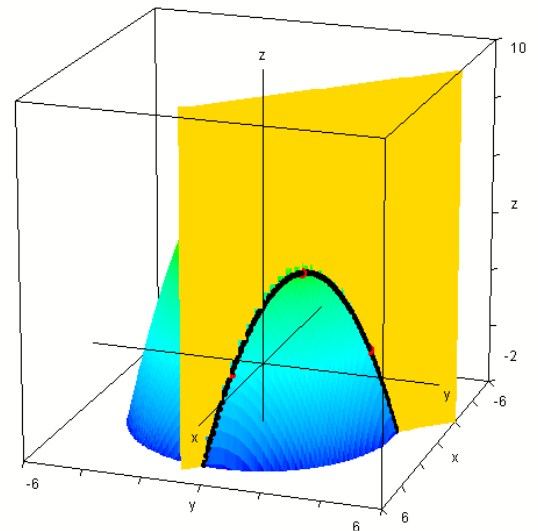
```
lagn_multi(x^2 + y^2 + z^2, [x^2 + y^2 + 2*z = 16, x + y = 4], [x, y, z])
```

x	y	z	m1	m2	Value
2	2	4	-4	12	24
$\sqrt{3} + 2$	$2 - \sqrt{3}$	1	-1	0	15
$2 - \sqrt{3}$	$\sqrt{3} + 2$	1	-1	0	15

It is surprising that DERIVE is unable to solve the equation in case of the root given in the function while TI-Nspire and even “old” TI-92” don’t have any problems.

Taking the square of the distance we receive the same result – we must take the square root of the “Values” to obtain the distances.

The plot shows the paraboloid, the plane and their intersection curve. The task is to find the points on this curve with minimum or maximum distance from the origin. The points are plotted in red color.



MATHEMATICA can solve the equation with the root:

```
In[4]:= Solve[{x / Sqrt[x^2 + y^2 + z^2] == 2 p * x + q,
y / Sqrt[x^2 + y^2 + z^2] == 2 p * y + q,
z / Sqrt[x^2 + y^2 + z^2] == 2 p, x^2 + y^2 + 2 z == 16, x + y == 4},
{x, y, z, p, q}]
```

```
Out[4]:= {{q -> 0, p -> 1/30 * Sqrt[1 + (2 - Sqrt[3])^2 + (2 + Sqrt[3])^2], z -> 1, x -> 2 - Sqrt[3],
```

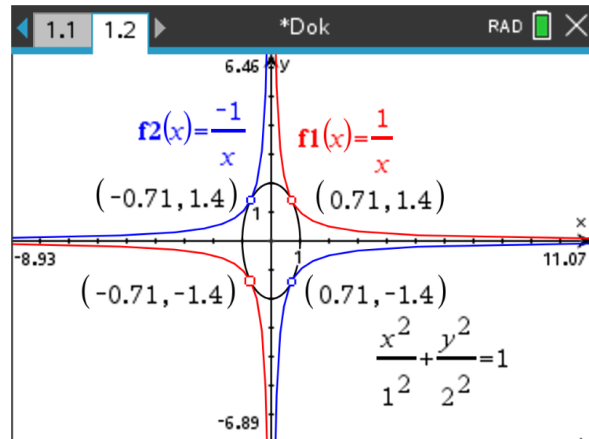
```
y -> 2 + Sqrt[3]}, {q -> 0, p -> 1/30 * Sqrt[1 + (2 - Sqrt[3])^2 + (2 + Sqrt[3])^2], z -> 1,
```

```
x -> 2 + Sqrt[3], y -> 2 - Sqrt[3]}, {q -> -Sqrt[3]/2, p -> 1/Sqrt[6], z -> 4, x -> 2, y -> 2}}]
```

3rd problem:

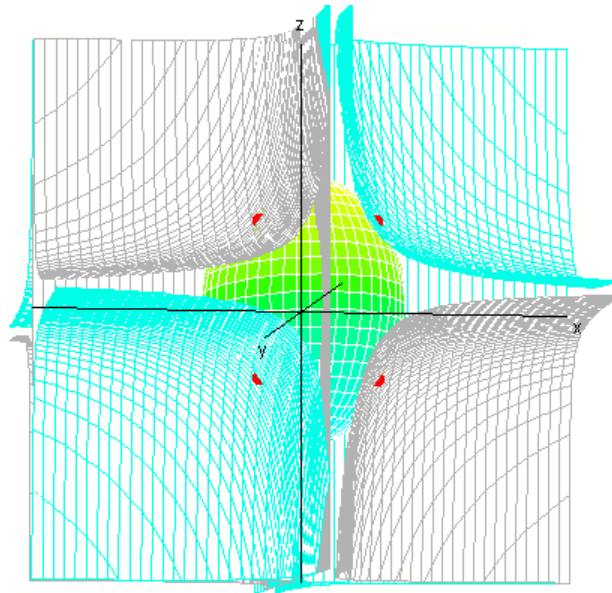
For which points on the ellipse $4x^2 + y^2 = 4$ is the product of their coordinates a maximum or a minimum?

Answers: $\left(\pm\frac{\sqrt{2}}{2}, \pm\sqrt{2}\right)$



This is the graphic representation of the fourth example shown on the TI-Nspire screen above: We can see eight solution points on the surface of an ellipsoid, which is touched by two “optimized”

surfaces $z(x,y) = \frac{c}{8xy}$ with $c = \pm 64\sqrt{3}$ (Maximum and Minimum)-

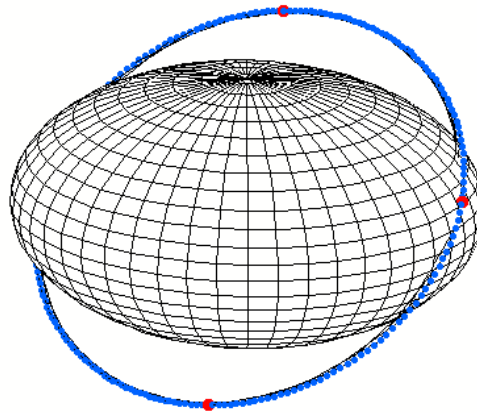


Finally, the last example with an ellipsoid to generate the optimum. The side conditions define the intersection curve between the sphere $([0,0,0]; r = 1)$ and a plane $(x + 2y + 3z = 0)$ passing the origin.

$$\text{lagr_multi}(x^2 + 2 \cdot y^2 + 3 \cdot z^2, [x^2 + y^2 + z^2 = 1, x + 2 \cdot y + 3 \cdot z = 0], [x, y, z])$$

x	y	z	m1		m2		Value
$\frac{\sqrt{(63 \cdot \sqrt{2} + 91)}}{14}$	$-\frac{\sqrt{(70 - 42 \cdot \sqrt{2})}}{14}$	$-\frac{\sqrt{(35 - 21 \cdot \sqrt{2})}}{14}$	$\frac{3 \cdot \sqrt{2}}{7}$	$\frac{12}{7}$	$\frac{\sqrt{(133 - 21 \cdot \sqrt{2})}}{49}$	$\frac{12}{7}$	$\frac{3 \cdot \sqrt{2}}{7}$
$\frac{\sqrt{(63 \cdot \sqrt{2} + 91)}}{14}$	$\frac{\sqrt{(70 - 42 \cdot \sqrt{2})}}{14}$	$\frac{\sqrt{(35 - 21 \cdot \sqrt{2})}}{14}$	$\frac{3 \cdot \sqrt{2}}{7}$	$\frac{12}{7}$	$\frac{\sqrt{(133 - 21 \cdot \sqrt{2})}}{49}$	$\frac{12}{7}$	$\frac{3 \cdot \sqrt{2}}{7}$
$-\frac{\sqrt{(91 - 63 \cdot \sqrt{2})}}{14}$	$-\frac{\sqrt{(42 \cdot \sqrt{2} + 70)}}{14}$	$-\frac{\sqrt{(21 \cdot \sqrt{2} + 35)}}{14}$	$\frac{3 \cdot \sqrt{2}}{7}$	$\frac{12}{7}$	$\frac{\sqrt{(21 \cdot \sqrt{2} + 133)}}{49}$	$\frac{3 \cdot \sqrt{2}}{7}$	$\frac{12}{7}$
$-\frac{\sqrt{(91 - 63 \cdot \sqrt{2})}}{14}$	$\frac{\sqrt{(42 \cdot \sqrt{2} + 70)}}{14}$	$\frac{\sqrt{(21 \cdot \sqrt{2} + 35)}}{14}$	$\frac{3 \cdot \sqrt{2}}{7}$	$\frac{12}{7}$	$\frac{\sqrt{(21 \cdot \sqrt{2} + 133)}}{49}$	$\frac{3 \cdot \sqrt{2}}{7}$	$\frac{12}{7}$

The plot shows the intersection circle together with the smallest ellipsoid of form $x^2 + 2y^2 + 3z^2 = c$ and the tangent points (red). The “free” red points are the tangent points of the largest ellipsoid.



It is a nice calculation to find the parameter form of this circle:

$$\left[\begin{array}{c} \frac{3 \cdot \cos(v)}{\sqrt{(4 \cdot \sin(v) \cdot \cos(v) + 3 \cdot \sin^2(v) + 10)}}, \frac{3 \cdot \sin(v)}{\sqrt{(4 \cdot \sin(v) \cdot \cos(v) + 3 \cdot \sin^2(v) + 10)}}, - \\ \frac{\cos(v)}{\sqrt{(4 \cdot \sin(v) \cdot \cos(v) + 3 \cdot \sin^2(v) + 10)}} - \frac{2 \cdot \sin(v)}{\sqrt{(4 \cdot \sin(v) \cdot \cos(v) + 3 \cdot \sin^2(v) + 10)}} \end{array} \right]$$

DNL References: DNL69, DNL112

The next function appearing in Bhuvanesh’s chapter “Optimization” is a tool for solving linear programming problems, it is the Simplex Method.

As I had to teach basics of Operations Research for several years, I have a rich collection of more or less complex linear programming problems. I tested the function on V200 using some small problems similar to this one presented by Bhuvanesh in his function description (next page). They all worked.

So, I started transferring the code to TI-Nspire. All the problems worked properly. Then I selected larger problems with not only \leq -constraints. If there was a \geq -inequality I multiplied by -1 and changed it to a \leq -inequality. If the constraint was an equation, I split it in two inequalities (see problems 1, 2 and 3 on pages 35, 36).

But then, applying the function on the next problem, the function turned out to be not reliable. And then I found more examples where the function did not work as expected. I am quite sure that I transported the TI-92 code correctly to TI-Nspire, because I received the same (wrong) results on my old TI-92. What to do??

Speaking about TI-92 I remembered that we had a great Simplex program for TI-92 long ago in DNL#26 from 1997 (!!). I started transferring this program into TI-Nspire-code adapting the input procedure of the parameters – and I was lucky that it finally worked without any problems.

See my results on page 37!!

I will start with Bhuvanesh’s Simplex function (part of his mathtool- library).

Simplex(obj,cmat) returns the minimum for the linear programming problem with the objective function given by **obj** and the constraints given by **cmat**

Needs: ListSubt, Select, SimpStep, Sort

Example: Minimize $f = 5x - 3y - 8z$ subject to the constraints $\{2x + 5y - z \leq 1, -2x - 12y + 3z \leq 9, -3x - 8y + 2z \leq 4, x \geq 0, y \geq 0, z \geq 0\}$

$\text{Simplex}(\{5, -3, -8, 0, 0, -1, 0\}, [[2, 5, -1, 1, 0, 0, 1], [-2, -12, 3, 0, 1, 0, 0, 9], [-3, -8, 2, 0, 0, 1, 0, 4]]) \Rightarrow \{-121, \text{vars} = \{0, 3, 14, 0, 3, 0\}\}$

Note: There are two widely used solution techniques for linear programming:

simplex methods and interior point methods. While simplex methods move from vertex to vertex along the edges of the boundary of the feasible region, interior point methods visit points within the interior of the feasible region. The Simplex function currently uses the tableau form of the simplex method.

The input list is obtained in the above example by setting $5x - 3y - 8z - f = 0$, so that f has a coefficient of -1 . You can also enter a shortened form of the list as $\{5, -3, -8\}$.

The rows in **cmat** give the coefficients of the original variables and the "slack" variables in the constraints. There is a slack variable for each inequality constraint. A slack variable allows you to turn an inequality constraint into an equality constraint. So, for example, the first constraint in the above example would become $2x + 5y - z + s1 = 1$. The final result displays the minimum of f

and the values of the original and slack variables at the minimum. So, for the above example, f has a minimum at -121 , where $\{x=0, y=3, z=14, s1=0, s2=3, s3=0\}$. To find the maximum rather than the minimum, simply find the minimum of $-f$ and flip the sign of f in the result.

```

F1 Control F2 I/O F3 Var F4 Find... F5 Mode
:simplex(obj,mat)
:Func
:@Simplex(obj,mat) returns the minimum f
or the linear programming problem with
objective function given by obj and con
straints given by mat
:Local nbvars,bvars,rd,ii,tmp,tmp2,mat:
dim(mat)->rd:rd[2]-rd[1]-2->tmp
:@Set up initial tableau
:If dim(obj)->rd[2] Then:augment(obj,seq
(0,ii,1,rd[2]-dim(obj)))>obj:1+obj[rd[2]
-1]:EndIf

```

```

F1 Control F2 I/O F3 Var F4 Find... F5 Mode
-1]:EndIf
:augment(mat,{obj})->mat
:seq(ii,ii,1,tmp)->nbvars
:mathtool\listsubt(seq(ii,ii,1,rd[2]-2),
nbvars)->bvars
:Loop
:If !!(when(mat[rd[1]+1,ii]>=0,1,0,0),ii,1
,rd[2])=1:Return (-mat[rd[1]+1,rd[2]],v
ars:=mat\list((mathtool\sort(augment(aug
ment(list\mat(bvars,1),subMat(mat,1,rd[
2],rd[1])):seq((nbvars[i],0),ii,1,dim(
nbvars))))))T[2]))

```

```

F1 Control F2 I/O F3 Var F4 Find... F5 Mode
:@Select non-basic var to make basic
:min(mat\list(mat[rd[1]+1]))>tmp
:For ii,1,rd[2]-1:If mat[rd[1]+1,ii]=tmp
:Exit:EndFor @Could have used Σ(when(...
),ii,1,rd[2]-1), but that could give inc
orrect results
:augment(bvars,(ii))->bvars
:mathtool\listsubt(nbvars,(ii))->nbvars
:@Select basic var to make non-basic
:mat->mat
:mat\list(mat[rd[2]])/(<mat\list(mat[ii
]))>tmp

```

```

F1 Control F2 I/O F3 Var F4 Find... F5 Mode
])>tmp
:min(mathtool\select(tmp,"#>0"))>tmp2
:For ii,1,rd[1]:If tmp[ii]=tmp2:Exit:End
:For @Could have used Σ(when(...),ii,1,rd[
2]-1), but that could give incorrect re
sults
:augment(nbvars,(bvars[ii]))>nbvars
:mathtool\listsubt(bvars,(bvars[ii]))>bv
ars
:@Pivoting to eliminate new basic var
:mathtool\simpstep(mat\list(mat[rd[1]+1
]),subMat(mat,1,1,rd[1]),bvars)->mat

```

```

F1 Control F2 I/O F3 Var F4 Find... F5 Mode
:For ii,1,rd[1]:If tmp[ii]=tmp2:Exit:End
:For @Could have used Σ(when(...),ii,1,rd[
2]-1), but that could give incorrect re
sults
:augment(nbvars,(bvars[ii]))>nbvars
:mathtool\listsubt(bvars,(bvars[ii]))>bv
ars
:@Pivoting to eliminate new basic var
:mathtool\simpstep(mat\list(mat[rd[1]+1
]),subMat(mat,1,1,rd[1]),bvars)->mat
:EndLoop
:EndFunc

```

```

F1 Algebra F2 Calc F3 Other F4 PrgmIO F5 Clean Up
simplex({5, -3, -8}, [[2, 5, -1, 1, 0, 0, 1],
[-2, -12, 3, 0, 1, 0, 0, 9], [-3, -8, 2, 0, 0, 1, 0, 4]])
(-121 vars = {0 3 14 0 3 0})
1,0,0,9;-3,-8,2,0,0,1,0,4])

```


Example 1: A Standard Maximum Problem

Capacity Optimization: A firm manufactures four products A, B, C and D, each of which must be processed through machines M1, M2 and M3. The following table - the "Input Table" - summarizes labour hour requirements per unit for each product on each machine. Also presented are weekly labour hours capacities b_i on each machine for these three machines and the profit margins for the products.

	A	B	C	D	b_i
M1	0	1	2	0	50
M2	2	0	4	2	80
M3	4	4	0	1	120
profit/unit	1.50	2.00	3.00	2.50	

It will take 4 hours on machine 2 to produce one unit of product C.

Maximize $z = 1.5x_1 + 2x_2 + 3x_3 + 2.5x_4$ under the restrictions:

$$(1) \quad 0 \cdot x_1 + 1 \cdot x_2 + 2 \cdot x_3 + 0 \cdot x_4 \leq 50 \quad (\text{for M1})$$

$$(2) \quad 2 \cdot x_1 + 0 \cdot x_2 + 4 \cdot x_3 + 2 \cdot x_4 \leq 80 \quad (\text{for M2})$$

$$(3) \quad 4 \cdot x_1 + 4 \cdot x_2 + 0 \cdot x_3 + 1 \cdot x_4 \leq 120 \quad (\text{for M3})$$

$$(4) \quad x_1, x_2, x_3, x_4, \geq 0 \quad \dots \dots \dots \text{Nonnegativity constraints}$$

Interpretation of the result:

Maximum = 140; produce 20 units of B and 40 units of D; 30 hours remain on machine M1.

Example 2: Diet Problem (Mixed Maximum)

A special daily diet should contain no more than 20g of fat, at least 300g of protein and exactly 170g of carbohydrates. It should also contain as much fiber as possible. Four dishes (X1 – X4) are available, the composition of which is described by the following table (in gram).

	X1	X2	X3	X4
Weight	100	100	100	100
Fat	12	3	17	2
Protein	50	20	30	40
Carbohydrates	30	15	20	20

In what quantities do the meals have to be given in order to fulfill the diet plan?

Maximize $z = 100x_1 + 100x_2 + 100x_3 + 100x_4$

$$(1) \quad 12x_1 + 3x_2 + 17x_3 + 2x_4 \leq 20$$

$$(2) \quad 50x_1 + 20x_2 + 30x_3 + 40x_4 \geq 300$$

$$(3) \quad 30x_1 + 15x_2 + 20x_3 + 20x_4 = 170$$

$$(4) \quad x_1, x_2, x_3, x_4 \geq 0$$

In order to apply `simplex()` we have to convert (2) into a \leq - restriction and to split (3) into two constraints (\leq and \geq).

Interpretation of the result:

Maximum = 900g, the diet consists of 200g of X2 and 700g of X4, 20g excess protein (allowed)

Example 3: One more Capacity Problem (3 products, 4 restrictions)

$$\begin{aligned} \text{Maximize} \quad & z = 3x_1 + 6x_2 + 4x_3 \\ (1) \quad & 3x_1 + 4x_2 + x_3 \leq 400 \\ (2) \quad & 4x_1 + 2x_3 \leq 300 \\ (3) \quad & 4x_2 + 2x_3 \leq 400 \\ (4) \quad & x_1 + 2x_2 \leq 150 \\ (5) \quad & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Interpretation of the result:

Maximum = 780, production of 30, 55 and 90 units, respectively; 10 time units remain restr. (4)

<p>© Bhatt's example</p> $\text{simplex}\left(\{5, -3, -8\}, \begin{bmatrix} 2 & 5 & -1 & 1 & 0 & 0 & 0 & 1 \\ -2 & -12 & 3 & 0 & 1 & 0 & 0 & 9 \\ -3 & -8 & 2 & 0 & 0 & 1 & 0 & 4 \end{bmatrix}\right) \quad \text{"Goal} = -121, \text{ vars} = \{0, 3, 14, 0, 3, 0\}"$ <p>© Example 1</p> $\text{simplex}\left(\left\{-\frac{3}{2}, -2, -3, -\frac{5}{2}, 0, 0, 0, 1, 0\right\}, \begin{bmatrix} 0 & 1 & 2 & 0 & 1 & 0 & 0 & 0 & 50 \\ 2 & 0 & 4 & 2 & 0 & 1 & 0 & 0 & 80 \\ 4 & 4 & 0 & 1 & 0 & 0 & 1 & 0 & 120 \end{bmatrix}\right) \quad \text{"Goal} = -140, \text{ vars} = \{0, 20, 0, 40, 30, 0, 0\}"$ <p>© Example 2</p> $\text{simplex}\left(\{-100, -100, -100, -100, 0, 0, 0, 0, -1, 0\}, \begin{bmatrix} 12 & 3 & 17 & 2 & 1 & 0 & 0 & 0 & 0 & 20 \\ -50 & -20 & -30 & -40 & 0 & 1 & 0 & 0 & 0 & -300 \\ 30 & 15 & 20 & 20 & 0 & 0 & 1 & 0 & 0 & 170 \\ -30 & -15 & -20 & -20 & 0 & 0 & 0 & 1 & 0 & -170 \end{bmatrix}\right) \quad \text{"Goal} = -900, \text{ vars} = \{0, 2, 0, 7, 0, 20, 0, 0\}"$ <p>© Example 3</p> $\text{simplex}\left(\{-3, -6, -4, 0, 0, 0, 0, -1, 0\}, \begin{bmatrix} 3 & 4 & 1 & 1 & 0 & 0 & 0 & 0 & 400 \\ 4 & 0 & 2 & 0 & 1 & 0 & 0 & 0 & 300 \\ 0 & 4 & 2 & 0 & 0 & 1 & 0 & 0 & 400 \\ 1 & 2 & 0 & 0 & 0 & 0 & 1 & 0 & 150 \end{bmatrix}\right) \quad \text{"Goal} = -780, \text{ vars} = \{30, 55, 90, 0, 0, 0, 10\}"$	<pre>simplex 11/21 Define LibPub simplex(obj,mat)= Func Local nbvars,bvars,rd,ii,tmp,tmp2,mat,res rd:=dim(mat):tmp:=rd[2]-rd[1]-2 If dim(obj)≠rd[2] Then obj:=augment(obj,seq(0,ii,1,rd[2]-dim(obj))) obj[rd[2]-1]:=1 EndIf mat:=(augment(mat,(list▶mat(obj)))^r):nbvars:=seq(ii,ii bvars:=mattool\listsubr(seq(ii,ii,1,rd[2]-2),nbvars) Loop rd[2] If ∏(when(mat[rd[1]+1,ii]≥0,1,0,0))=1 Then res:=mattool\sort(augment(augment(list▶mat(bvar Return "Goal = "&string(-mat[rd[1]+1,rd[2]])& ", var EndIf tmp:=min(mat▶list(mat[rd[1]+1])) For ii,1,rd[2]-1: If mat[rd[1]+1,ii]=tmp:Exit: EndIf bvars:=augment(bvars,{ii}): nbvars:=mattool\listsu tmp:=mat▶list(mat[rd[2]]) : tmp2:=min(mattool\sele mat▶list(mat[ii]) For ii,1,rd[1]: If tmp[ii]=tmp2:Exit: EndFor nbvars:=augment(nbvars,{bvars[ii]}): bvars:=mathtc mat:=mattool\simpstep(mat▶list(mat[rd[1]+1]),subM EndLoop EndFunc</pre>
--	---

Example 4: Primal Problems and their Dual

Working on linear programming problems one will come across the duality principle: it says, that to every maximum problem there exists its dual minimum problem and vice versa less than inequalities are exchanged by greater than constraints and vice versa.

I present a small example of a minimum problem and its dual on the next page. The mattool\simplex does not work for the primal problem, but it does for the dual one.

Then I asked the TI-Nspire version of Bruce Chaffee's TI-92 program from 1997. In the original program the user can enter the parameters in dialogue form. This is not possible in a convenient way with TI-Nspire,

The primal minimum problem must be entered as

$$\text{simplex}(\{450, 1000\}, \{0, 2, 0\}, \begin{bmatrix} 1 & 3 & 150 \\ 1 & 0 & 30 \\ 0 & 1 & 20 \end{bmatrix}, \text{output option, MinMax})$$

Simplex Method – Duality

Primal: Minimum Problem

$$450x + 1000y = \text{Minimum}$$

$$x + 3y \geq 150$$

$$x \geq 30$$

$$y \geq 20$$

$$x, y \geq 0$$

Dual: Maximum Problem

$$150u + 30v + 20w = \text{Maximum}$$

$$u + v \leq 450$$

$$3u + w \leq 1000$$

$$u, v, w \geq 0$$

$$\text{mathtool}\backslash\text{simplex}\left(\left\{450,1000,0,0,0,1,0\right\},\begin{bmatrix} -1 & -3 & -1 & 0 & 0 & 0 & -150 \\ -1 & 0 & 0 & -1 & 0 & 0 & -30 \\ 0 & -1 & 0 & 0 & -1 & 0 & -20 \end{bmatrix}\right) \blacktriangleright \text{Goal} = 0, \text{vars} = \{0,0,-150,-30,-20\}$$

does not work!!

$$\text{mathtool}\backslash\text{simplex}\left(\left\{-150,-30,-20,0,0,-1,0\right\},\begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 450 \\ 3 & 0 & 1 & 0 & 1 & 0 & 1000 \end{bmatrix}\right) \blacktriangleright \text{Goal} = -53500, \text{vars} = \{1000/3,350/3,0,0,0\}$$

works!!

$$\text{mathtool}\backslash\text{simplex}\left(\left\{-1000,-1200,0,0,0,-1,0\right\},\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 40 \\ 200 & 600 & 0 & 1 & 0 & 0 & 12000 \\ 5 & 10 & 0 & 0 & 1 & 0 & 240 \end{bmatrix}\right) \blacktriangleright \text{Goal} = -41600, \text{vars} = \{32,8,0,800,0\}$$

$$\text{mathtool}\backslash\text{simplex}\left(\left\{-40,-12000,-240,0,0,-1,0\right\},\begin{bmatrix} -1 & -200 & -5 & -1 & 0 & 0 & -1000 \\ -1 & -600 & -10 & 0 & -1 & 0 & -1200 \end{bmatrix}\right) \blacktriangleright \text{Goal} = -42000, \text{vars} = \{900,1/2,0,0,0\}$$

$$\text{mathtool}\backslash\text{simplex}\left(\left\{-1,-1,-1,0,0,0,-1,0\right\},\begin{bmatrix} 0 & 4 & 10 & 1 & 0 & 0 & 0 & 300 \\ 10 & 3 & 0 & 0 & 1 & 0 & 0 & 500 \\ 4 & 5 & 2 & 0 & 0 & 1 & 0 & 350 \end{bmatrix}\right) \blacktriangleright \text{Goal} = -89, \text{vars} = \{41,30,18,0,0,0\}$$

$$\text{mathtool}\backslash\text{simplex}\left(\left\{-300,-500,-350,0,0,0,-1,0\right\},\begin{bmatrix} 0 & -10 & -4 & -1 & 0 & 0 & 0 & -1 \\ -4 & -3 & -5 & 0 & -1 & 0 & 0 & -1 \\ -10 & 0 & -2 & 0 & 0 & -1 & 0 & -1 \end{bmatrix}\right) \blacktriangleright \text{Goal} = -89, \text{vars} = \{2/25,3/50,1/10,0,0,0\}$$

output option can be *s* (for solution only), *i* (intermediate results), *d* (decision variables) or *b* (both).

Between the 2nd braces enter the number of \leq , \geq , and $=$ constraints in this order and in the matrix enter the coefficients of the \leq , \geq , and $=$ constraints in this order, too. The last argument is 1 for maximum and -1 for minimum.

See below another example for dual problems and compare with the results above (`mathtool\simplex`, problem 3 and 4). Problems 5 and 6 are correct – not reliable!! Bruce Chaffee’s tool is reliable (I hope!).

```
simplex({1000,1200},{3,0,0},[1 1 40;200 600 12000;5 10 240],s,1)
PRELIMINARY TABLEAU
[1 1 1 0 0 40
200 600 0 1 0 12000
5 10 0 0 1 240
-1000 -1200 0 0 0 0
0 0 0 0 0 0]
Press a Key
BOUNDED SOLUTION FOUND
Press a Key
X(1) = 32
X(2) = 8
S(1) = 0
S(2) = 800
S(3) = 0
P = 41600
In Case of Minimum: -41600
```

```
simplex({40,12000,240},{0,2,0},[1 200 5 1000;1 600 10 1200],s,-1)
PRELIMINARY TABLEAU
[1 200 5 -1 0 1 0 1000
1 600 10 0 -1 0 1 1200
40 12000 240 0 0 0 0 0
0 0 0 0 0 1 1 0]
Press a Key
BOUNDED SOLUTION FOUND
Press a Key
X(1) = 800
X(2) = 0
X(3) = 40
S(1) = 0
S(2) = 0
A(1) = 0
A(2) = 0
Press a key to Continue
P = -41600
In Case of Minimum: 41600
```

Spirograph Figures with Sampling

Rick Nungester, February 26, 2022

A Spirograph is a drawing tool made from a flat ring with gear teeth on its inner and outer edges. A wheel with matching gear teeth rotates around the inside or outside of the ring. A pen is inserted through the wheel at one of several distances from the center of the wheel. As the pen follows the wheel rotation attractive figures result. Example figures and the math involved are available online, for example at:

<https://en.wikipedia.org/wiki/Spirograph>

[https://de.wikipedia.org/wiki/Spirograph_\(Spielzeug\)](https://de.wikipedia.org/wiki/Spirograph_(Spielzeug))

My interest in Spirograph mathematics goes back to the early 1980s and my first home computer, a [Hewlett-Packard 2647A Intelligent Graphics Terminal](#). I started work with HP in 1977 and the terminal was a cosmetic reject unit I was fortunate enough to obtain for personal use. It was built around a screaming fast Intel 8080 CPU running at 2.5 MHz! A BASIC interpreter had to be loaded from magnetic tape after each power-on, and the BASIC included vector graphics commands! I was in heaven. I wrote a Spirograph program with a “points per wheel revolution” parameter and noticed some attractive images could be made by lowering the points per wheel revolution to fractional values such as 2.02 or 2.97. Fast forward to 2021 and I bought a Spirograph for use with my grandchildren. This renewed my interest in the mathematics, and I further developed them using Derive. This article is the result of those efforts.

Details are given in file Spirograph_Sampling.dfw but this summary gives an overview of the process.

1. Experiment with Spirograph images from the Design Guide included with the product. This verifies the equations being used are correct.
2. Extend the images to those that can't be physically made with the device. For example: The pen being farther from the center of the wheel than the wheel radius. A wheel rotating inside the ring with radius larger than the radius of the ring. A 1-tooth wheel.
3. Find a good image to sample (and this is key). Find a very dense image, something that has many wheel revolutions and fills an area with many lines in an understood way.
4. Sample that image by reducing the number of points per wheel revolution to 2 (a line), 3 (a triangle), 4 (a square), etc. Look at the resulting figure and understand what is happening to the polygon during the plot as the wheel makes its many revolutions.
5. Offset the sample value (points per wheel revolution) slightly to make the polygon rotate a known number of rotations clockwise (CW) or counterclockwise (CCW) during the many revolutions of the wheel around the ring. Choose a sample value that will return the pen to its starting position at the end of the plot.

That process results in the following figures, from file Spirograph_Sampling.dfw:

Rr = ring radius. Rw = wheel radius. Rp = pen radius on wheel.

Ring and wheel radii are proportional to (so chosen equal to) their number of gear teeth.

Initially the pen is centered left-right and as high as possible.

*** Both Inner and Outer Figures ***

Reduce the fraction <ring teeth>/<wheel teeth> to get <image points>/<wheel center revs about ring center>.

$$\text{ImgPts}(Rr, Rw) := \frac{\text{LCM}(Rr, Rw)}{Rw}$$

$$\text{WhCtrRevs}(Rr, Rw) := \frac{\text{LCM}(Rr, Rw)}{Rr}$$

*** Inner Figures ***

The wheel center moves CCW and the wheel rotates CW.

WhRevsIn = Wheel Revolutions about its center, required to complete an Inner figure.

$$X_{in}(Rr, Rw, Rp, t) := - (Rr - Rw) \cdot \text{SIN} \left(\frac{t}{\frac{Rr}{Rw} - 1} \right) + Rp \cdot \text{SIN}(t)$$

$$Y_{in}(Rr, Rw, Rp, t) := (Rr - Rw) \cdot \text{COS} \left(\frac{t}{\frac{Rr}{Rw} - 1} \right) + Rp \cdot \text{COS}(t)$$

$$\text{WhRevsIn}(Rr, Rw) := \text{WhCtrRevs}(Rr, Rw) \cdot \left| \frac{Rr}{Rw} - 1 \right|$$

Test image: 96 ring teeth, 60 wheel teeth, pen hole near the wheel rim.

96/60 = 8/5 so 8 image points, 5 wheel center revs about ring center, 3 wheel revs about its center.

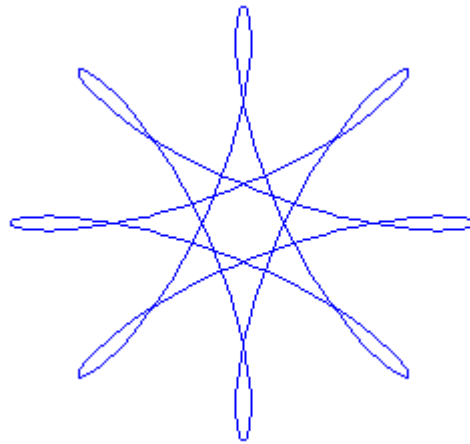
$$[\text{ImgPts}(96, 60), \text{WhCtrRevs}(96, 60), \text{WhRevsIn}(96, 60)] = [8, 5, 3]$$

PPWR = Points Per Wheel Revolution to plot.

Scale all plots to fit in a circle of radius 1.

$$\text{SpiroIn}(Rr, Rw, Rp, \text{PPWR}) := \text{VECTOR} \left(\left[\frac{X_{in}(Rr, Rw, Rp, t)}{Rr - Rw + Rp}, \frac{Y_{in}(Rr, Rw, Rp, t)}{Rr - Rw + Rp} \right], t, 0, \left(\text{WhRevsIn}(Rr, Rw) + \frac{1}{2 \cdot \text{PPWR}} \right) \cdot 2 \cdot \pi, \frac{2 \cdot \pi}{\text{PPWR}} \right)$$

SpiroIn(96, 60, 50, 100)



That plot agrees with what a real Spirograph generates.

*** Outer Figures ***

Wheel center moves CCW and the wheel rotates CCW.

$$X_{out}(R_r, R_w, R_p, t) := - (R_r + R_w) \cdot \text{SIN} \left(\frac{t}{\frac{R_r}{R_w} + 1} \right) - R_p \cdot \text{SIN}(t)$$

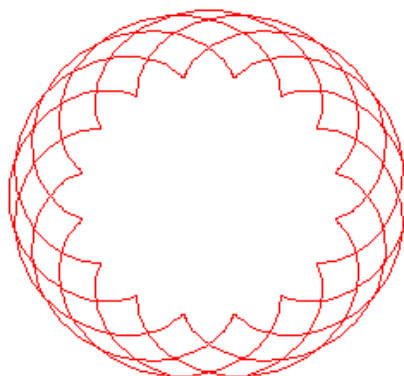
$$Y_{out}(R_r, R_w, R_p, t) := (R_r + R_w) \cdot \text{COS} \left(\frac{t}{\frac{R_r}{R_w} + 1} \right) + R_p \cdot \text{COS}(t)$$

$$\text{WhRevsOut}(R_r, R_w) := \text{WhCtrRevs}(R_r, R_w) \cdot \left(\frac{R_r}{R_w} + 1 \right)$$

$$\text{SpiroOut}(R_r, R_w, R_p, \text{PPWR}) := \text{VECTOR} \left(\left[\frac{X_{out}(R_r, R_w, R_p, t)}{R_r + R_w + R_p}, \frac{Y_{out}(R_r, R_w, R_p, t)}{R_r + R_w + R_p} \right], t, 0, \left(\text{WhRevsOut}(R_r, R_w) + \frac{1}{2 \cdot \text{PPWR}} \right) \cdot 2 \cdot \pi, \frac{2 \cdot \pi}{\text{PPWR}} \right)$$

Test image: 144 ring teeth, 45 wheel teeth, hole near the wheel rim.

`SpiroOut(144, 45, 40, 50)`



*** Generalize to lines, triangles, squares... rotating CCW or CW as they go around CCW ***

Npp = number of polygon points (2, 3, 4... = line, triangle, square...)

Npr = number of polygon revolutions (+ = CCW, - = CW, can be fractional)

Nwr = number of wheel revolutions about its center.

PPWRO = Points Per Wheel Rev about its center, for Outer figures.

PPWRI = Points Per Wheel Rev about its center, for Inner figures.

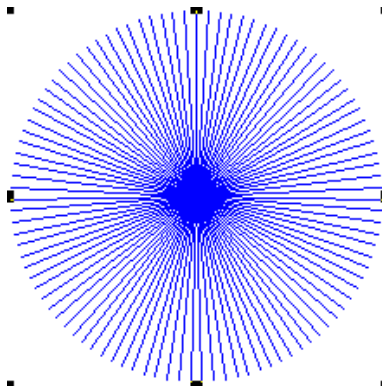
$$\text{PPWRO}(Npp, Npr, Nwr) := Npp \cdot \left(1 - \frac{Npr}{Nwr}\right)$$

$$\text{PPWRI}(Npp, Npr, Nwr) := Npp \cdot \left(1 + \frac{Npr}{Nwr}\right)$$

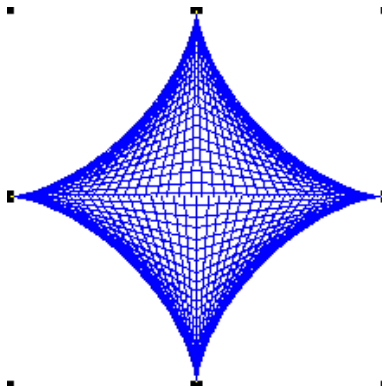
$\text{SpiroSampOut}(Rr, Rr, Rr, Npp, Npr) := \text{SpiroOut}(Rr, Rr, Rr,$
 $\text{PPWRO}(Npp, Npr, \text{WhRevsOut}(Rr, Rr)))$

$\text{SpiroSampIn}(Rr, Rr, Rr, Npp, Npr) := \text{SpiroIn}(Rr, Rr, Rr, \text{PPWRI}(Npp,$
 $Npr, \text{WhRevsIn}(Rr, Rr)))$

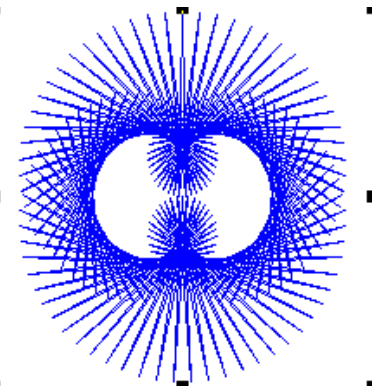
A collection of attractive images follows:



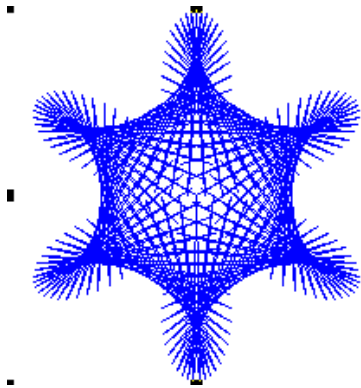
$\text{SpiroSampOut}(99, 1, 100, 2, 1)$



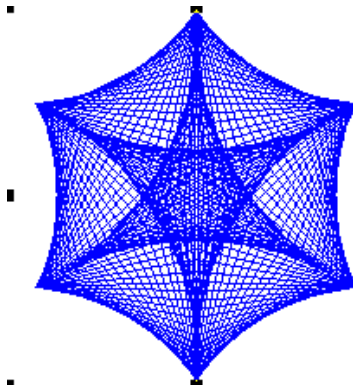
$\text{SpiroSampOut}(99, 1, 100, 2, -1)$



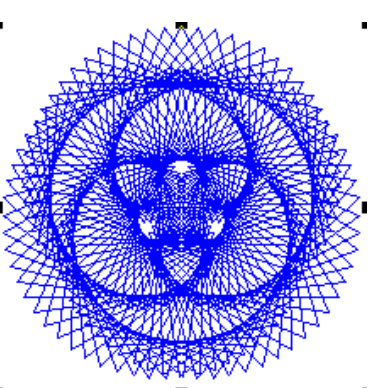
$\text{SpiroSampOut}(99, 1, 100, 2, 2)$



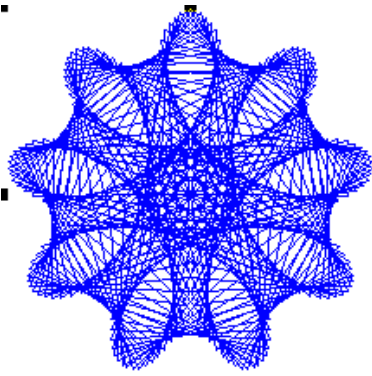
$\text{SpiroSampOut}(99, 1, 100, 2, -2)$



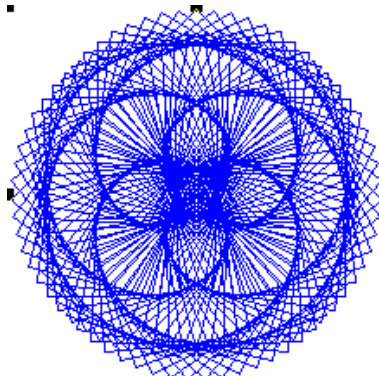
$\text{SpiroSampOut}(99, 1, 100, 3, -1)$



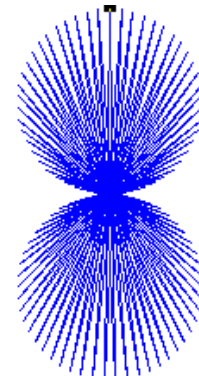
$\text{SpiroSampOut}(99, 1, 100, 3, 2)$



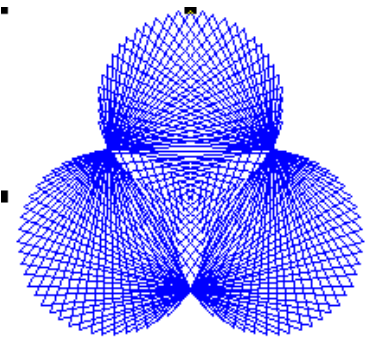
SpiroSampOut(99, 1, 100, 3, -2)



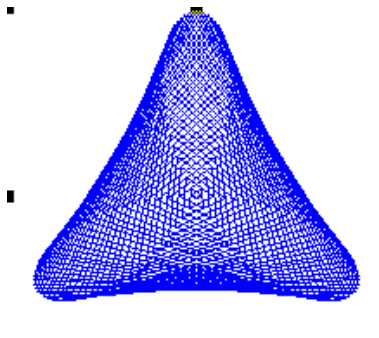
SpiroSampOut(99, 1, 100, 4, 2)



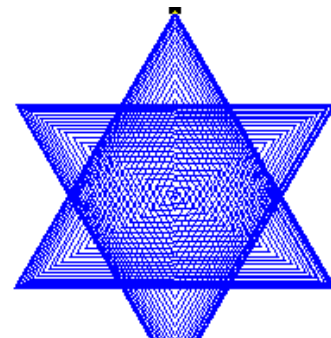
SpiroSampOut(1, 49, 50, 2, 0)



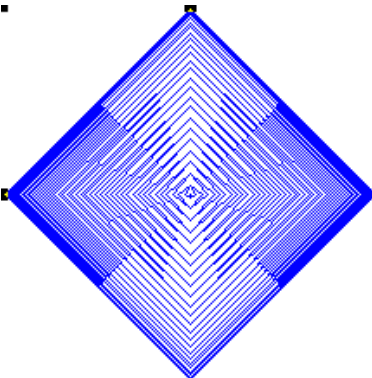
SpiroSampOut(1, 49, 50, 3, 0)



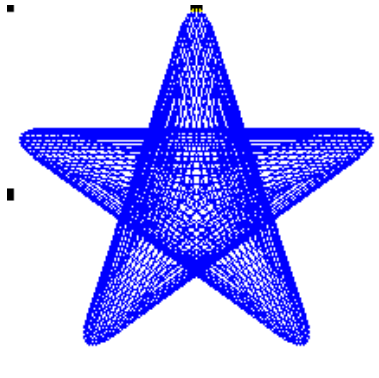
SpiroSampOut(1,49,50,3,1/3)



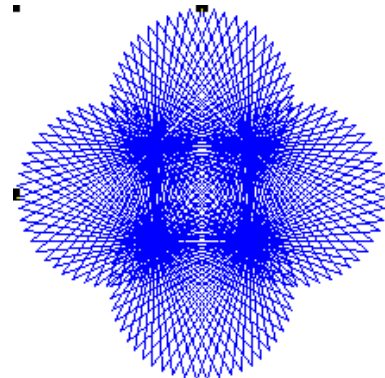
SpiroSampOut(1,49,50,3,1/2)



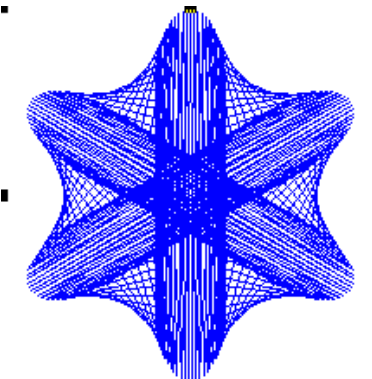
SpiroSampIn(1,49,50,4,1/2)



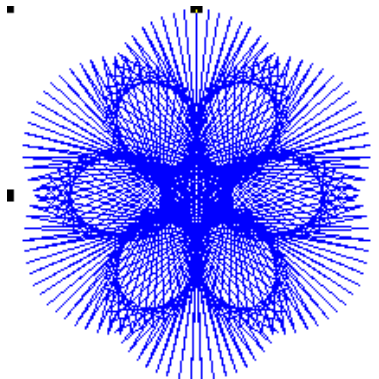
SpiroSampIn(151,50,101,2,1/2)



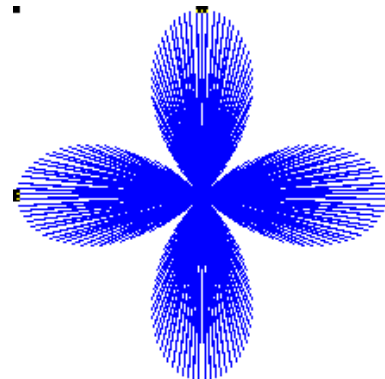
SpiroSampIn(151,50, 101, 2, -1)



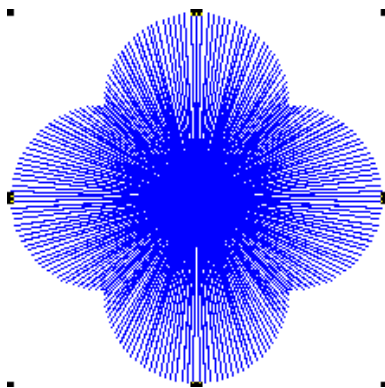
SpiroSampIn(151,50,101,3,1/3)



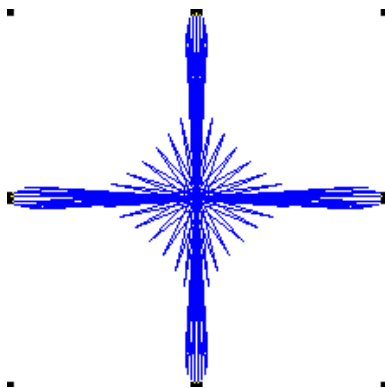
SpiroSampIn(151,50,101,3,-1)



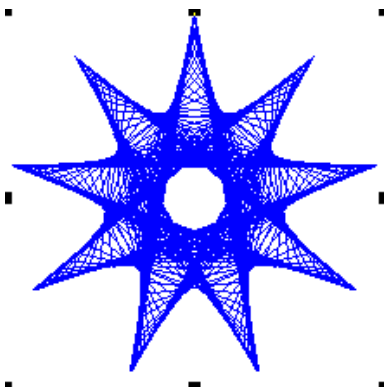
SpiroSampIn(201,100,101,2,1/2)



SpiroSampln(201,100,101 4, 0)



SpiroSampln(101,50,25,2,-3/2)



SpiroSampln(201, 100, 50, 3, -2)

Our communication when preparing the contribution for the DNL follows:

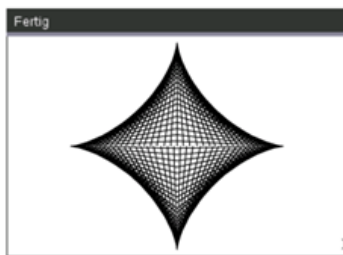
Dear Rick,

I could not resist transferring your Spirograph functions and pictures on the TI-NspireCAS.

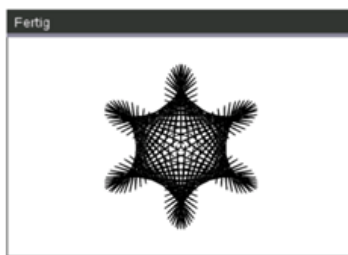
I found out that sometimes the spsampout- and spsampin-figures with same parameters look almost the same. See my two pictures (middle of first row and right of second row).

Josef

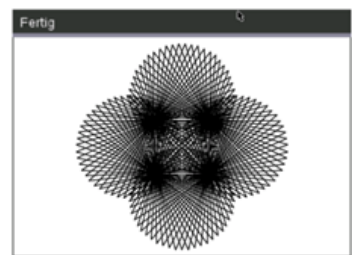
spout(99,1,100,2.02,100)



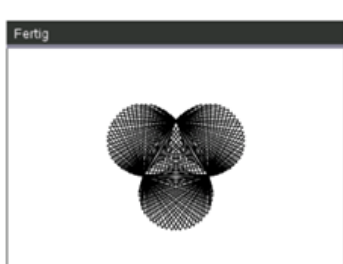
spsampout(99,1,100,2,-2,80)



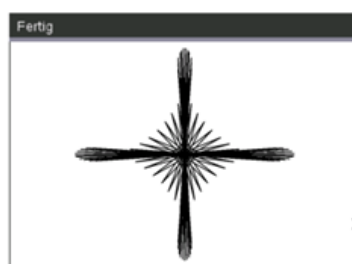
spsampin(151,50,101,2,-1,100)



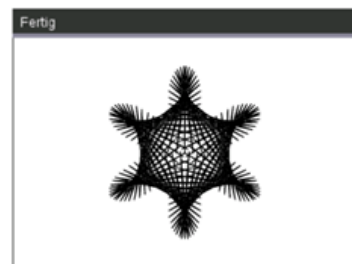
spsampout(1,49,50,3,0,70)



spsampin(101,50,25,2,-3/2,100)



spsampin(99,1,100,2,-2,80)



On 1/19/2022 12:17 PM, Rick Nungester wrote:

Hello Josef,

Spout(99,1,100...) and Spin(99,1,100...) are both a big ring with a tiny wheel making many big circles as the wheel moves around the ring once. But there are details in their difference.

SpiroOut(99,1,100, 50) is shown in my Derive file and makes exactly 100 circles (99 + 1 due to the wheel going around the ring counterclockwise once as it draws 99 counterclockwise circles) with the pen going exactly to the origin (ring radius 99 + wheel radius 1 = pen radius 100).

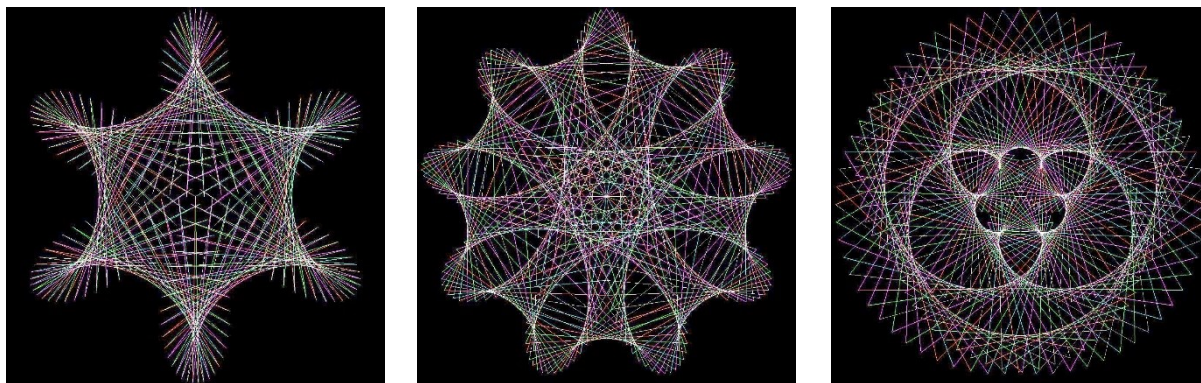
$\text{SpiroIn}(99,1,100, 50)$ makes *98 circles* (99 - 1 due to the wheel going around the ring counterclockwise once as it draws 99 *clockwise* circles) with each circle extending *past the origin by 2* (ring radius 99 - wheel radius 1 = 98 away from the origin and this figure uses a pen radius of 100). Plot it in my Derive file and you see a little hole in the middle because the pen always misses touching the origin. So this is a little different than desired, but not by very much. Sampling it hides the issue.

My 1980s work was done in BASIC and transferred to Derive in November 2021. Parameters Npp and Npr were added then, along with PPWRO() and PPWRI(), to avoid the logic and math related to: Wheel inside or outside the ring? How many wheel revolutions is that? Then what is the desired sampling rate?

Josef,

One more thing -- I wrote a Pascal program November 2021 to help me find good images before I put them in Derive. It allows changing parameters at the command line quickly and iterating to find good images. I just did screen captures on the 22 images in the Derive file, from that program (then batch cropped and bordered). Because they are mostly black, they compress well and all 22 are attached. I don't think they should go with the DUG article but I wanted to share them. This is what I saw before I put the images in Derive. I really like the larger size and random coloring.

Thanks again,
Rick



Rick,

I added two commands in your DERIVE file to receive colored graphs. Plotting takes some time – but it works.

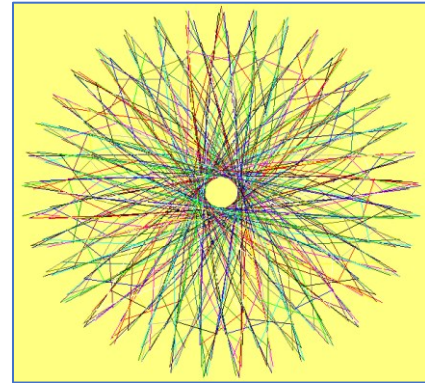
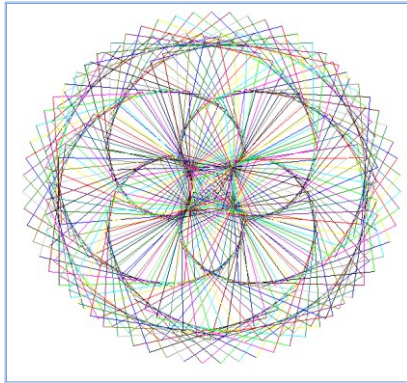
I'll try to make plotting faster.

The “trick” is to not only plot the figure as a polygon consisting of connected points, but to use the points to produce a set of segments (which are plotted separately).

When Changing Colors in the Plot Window is activated then you get colored graphs.

Best regards

Josef



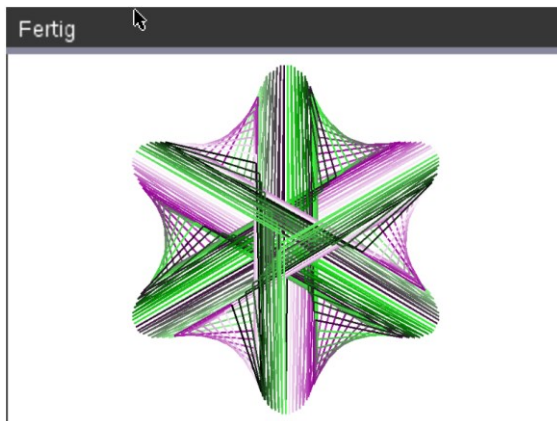
The Spirograph with TI-Nspire:

<pre> spout(99,1,100,2,100) Fertig ppwro(npp,nnpr,nwr):=npp*(1-nnpr/nwr);ppwri(npp,nnpr,nwr):=npp*(1+n Fertig sampout_x(rr,rw,rp,npp,nnpr,f):=spout_x(rr,rw,rp,ppwro(npp,nnpr,whr Fertig sampout_y(rr,rw,rp,npp,nnpr,f):=spout_y(rr,rw,rp,ppwro(npp,nnpr,whr Fertig DrawPoly sampout_x(99,1,100,2,1,100),sampout_y(99,1,100,2,1,100) Fertig DrawPoly sampout_x(99,1,100,2,-1,100),sampout_y(99,1,100,2,-1,100) Fertig DrawPoly sampout_x(99,1,100,2,-2,100),sampout_y(99,1,100,2,-2,100) Fertig sampin_x(rr,rw,rp,npp,nnpr,f):=spin_x(rr,rw,rp,ppwri(npp,nnpr,whrev) Fertig sampin_y(rr,rw,rp,npp,nnpr,f):=spin_y(rr,rw,rp,ppwri(npp,nnpr,whrev) Fertig DrawPoly sampin_x(151,50,101,3,-1,100),sampin_y(151,50,101,3,-1,100) Fertig spsampin(151,50,101,3,1/3,100) Fertig spsampin(151,50,101,3,-1/3,100) Fertig </pre>	<pre> spsampin 3/3 Define spsampin(rr,rw,rp,npp,nnpr,f)= Prgm xvals:=approx(sampin_x(rr,rw,rp,npp,nnpr,f)) yvals:=approx(sampin_y(rr,rw,rp,npp,nnpr,f)) DrawPoly xvals,yvals EndPrgm spsampout 3/3 Define spsampout(rr,rw,rp,npp,nnpr,f)= Prgm xvals:=approx(sampout_x(rr,rw,rp,npp,nnpr,f)) yvals:=approx(sampout_y(rr,rw,rp,npp,nnpr,f)) DrawPoly xvals,yvals EndPrgm </pre>
--	---

We plot the figure as a polygon. (We could give one color using the setcolor command).

	<pre> 1.1 1.2 1.3 *spirograph RAD 1/15 spiro.py from ti_system import * from ti_draw import * from math import * xv=recall_list("xvals") yv=recall_list("yvals") for i in range(0,len(xv)-1): **r=abs(int(fmod(255-2*i,255))) **g=abs(int(fmod(255-5*i,255))) **b=abs(int(fmod(255-2*i,255))) **set_color(r,g,b) **draw_line(xv[i],yv[i],xv[i+1],yv[i+1]) </pre>
--	--

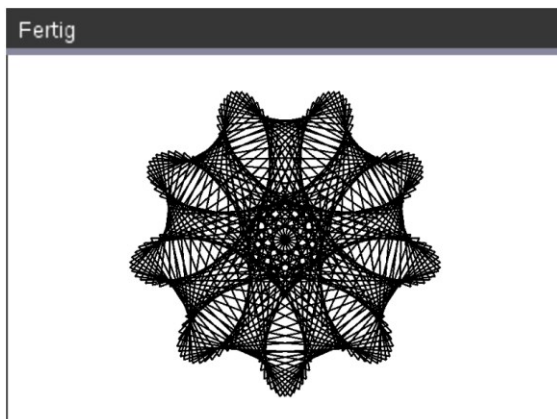
Just for fun I wanted to turn the wheels with Python



```

1.1 1.2 1.3 *spiograph RAD
spsampin(151,50,101,3,1/3,100) Fertig
spsampout(1,49,50,3,0,70) Fertig
spsampin(101,50,25,2,-3/2,100) Fertig
spsampout(1,49,50,3,0,70) Fertig
spsampout(99,1,100,3,-2,90)

```



```

1.2 1.3 1.4 *spiograph RAD
spiro.py 8/15
from ti_system import *
from ti_draw import *
from math import *
xv=recall_list("xvals")
yv=recall_list("yvals")
for i in range(0,len(xv)-1):
    r=abs(int(fmod(255-2*i,255)))
    g=abs(int(fmod(3*i,255)))
    b=abs(int(fmod(100,1)))
    set_color(r,g,b)
    draw_line(xv[i],yv[i],xv[i+1],yv[i+1])

```

A completely unicolor figure in black and the same figure with changing colors.

