The role of technology for the development and use of mathematical understanding

Didactic comment:

Favorable for the stimulation of elaborative learning is that the given problem is also interesting for the students.

The most topical problem at the moment is the Corona pandemic. So many data are published and even politicians try to explain the exponential growth – not always successful. A great advantage of technology in problem solving is the use of larger amounts of data

This current topic is very suitable for finding answers to two questions:

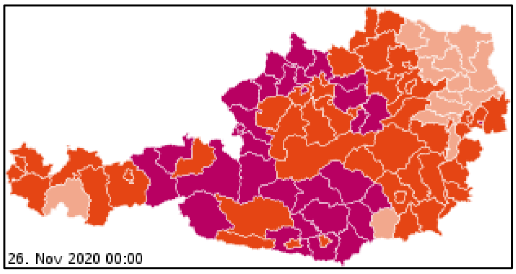
1. **What can we learn from Mathematics about Corona?**

But just as important for the goals of mathematics education is the inversion of the question:

2. **What can we learn from Corona about mathematics?**

Example 1: The Corona pandemic: Exponential growth – yes or no?

The development of the number of Corona-infected persons in Austria since the beginning of the pandemic.



Various websites offer data on the number of infected persons in Austria since the beginning of the Corona pandemic.

Source: http://www.covidanalysen.at

Task:

Part 1: What can we learn from Mathematics about Corona?

1. Describe the growth process and investigate the development of the total number and the daily new infected persons especially in October and November. Can be observed periods with an exponential growth?
2. Use two strategies to strengthen your assumptions:
   - Calculate regression functions and draw the graphs
   - Investigate also the graph of the total number of infected persons when using a logarithmic scale. What graph do you expect if you guess an exponential growth?
3. Describe the development of the lockdown periods
4. Calculate the „doubling time“ of periods with exponential growth.

Part 2: What can we learn from Corona about mathematics?

5. What is the mathematical relationship between the total number of infected persons and the number of the daily new infected persons? Investigate this relationship in a period of exponential growth (e.g. end of October, beginning of November)
6. Prove the rules which you have used in Part 1.

Didactic comment:

In the classroom I would give such an example with more open questions. Students should at first discuss what tasks are interesting or possible.

Solutions

Part 1:

Tasks 1 and 2:

The given data are entered in a table.

	A	B	C
259	09.11.2020	158116	[illegible]
260	10.11.2020	164075	5959
261	11.11.2020	172980	8905
262	12.11.2020	182178	9198
263	13.11.2020	191325	9147
264	14.11.2020	198601	7276
265	15.11.2020	203612	5011
266	16.11.2020	207832	4220
267	17.11.2020	213972	6140
268	18.11.2020	221631	7659
269	19.11.2020	228520	6889
270	20.11.2020	234778	6258
271	21.11.2020	241294	6516
272	22.11.2020	246086	4792
273	23.11.2020	249157	3071
274	24.11.2020	253649	4492
275	25.11.2020	259471	5822

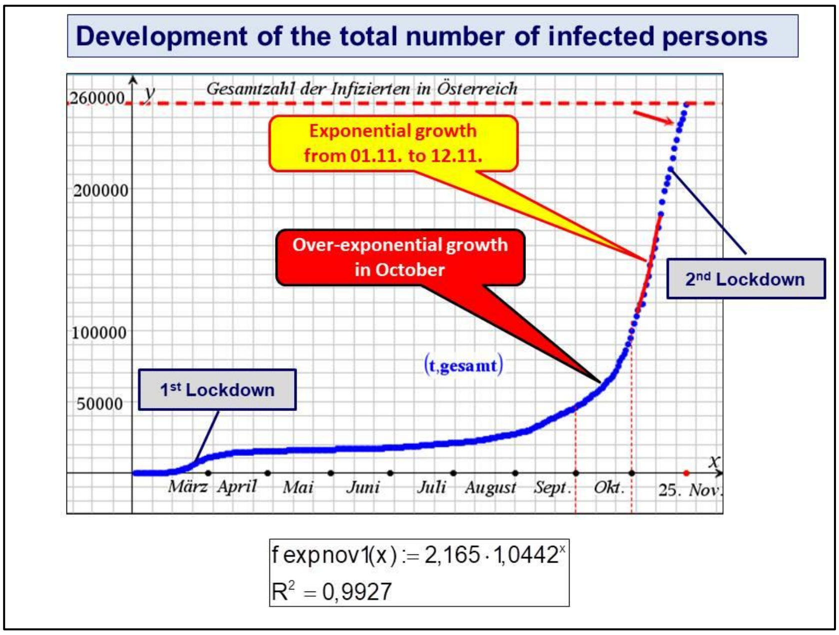
Total sum of infected persons

Daily new infected persons

This representation is not useful to decide the question "Exponential growth – yes or no?"

The best representation for interpreting the situation and coming to assumptions about the development of the Corona pandemic is the graphic representation.

Interpreting the graph of the total number of infected persons



Assumptions:

In October we assume an over-exponential growth, the growth rate increases.

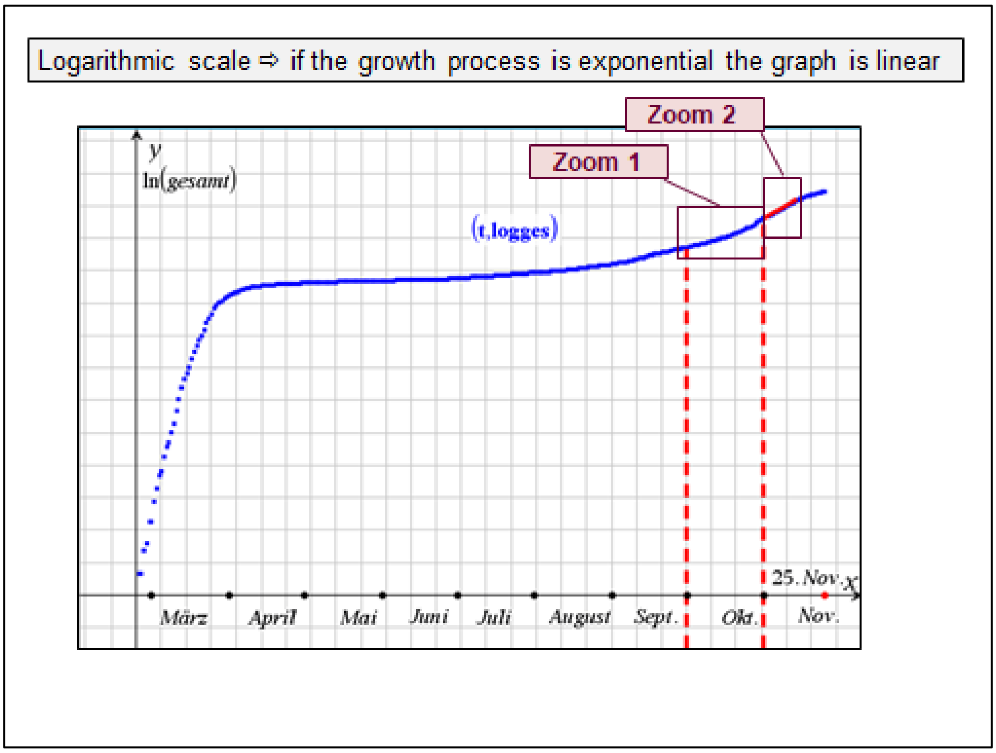
In November it could be a linear growth or an exponential growth.

Two strategies are thinkable to strengthen our assumptions:

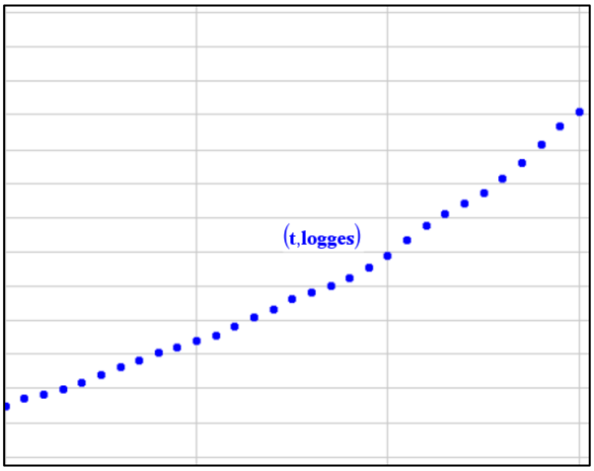
Strategy 1: With the technology tool it is possible to calculate **regression functions:**

For beginning of November we get an exponential function “fexpnov1”. The coefficient of determination R^2 of the exponential regression function strengthens the supposition.

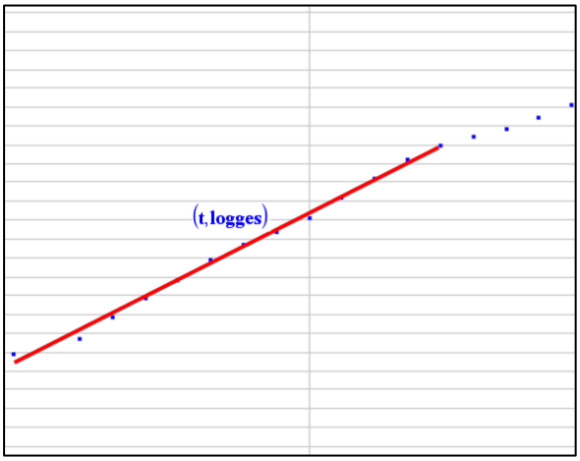
Strategy 2: If we would use a **logarithmic scale** for the function values of an exponential function the graph must be linear.



Interpreting the graph of the total pandemic does not result in a clear solution. More interesting is to zoom in certain intervals e.g. in October or November.



The graph is not linear in October, the curvature is negative ⇨ the growth is over-exponential

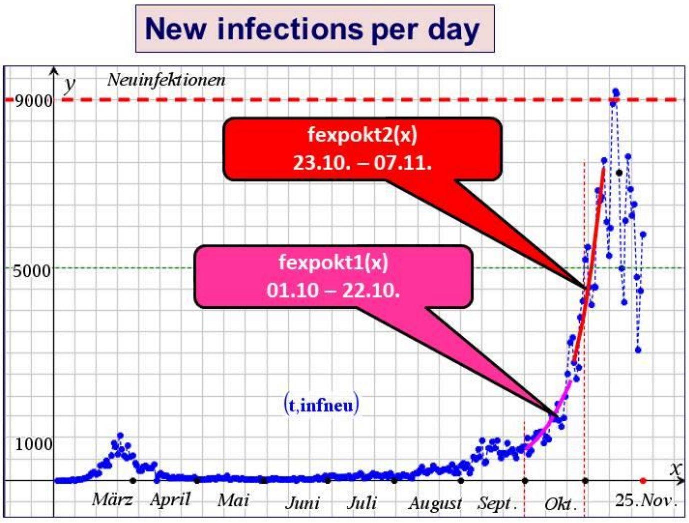


The graph at the beginning of November is linear. The linear regression function gives a good approximation.

$$f\,loglin(x) := 0{,}0439 \cdot x + 0{,}6083$$
$$R^2 = 0{,}993$$

Interpreting the graph of number of daily new infected persons.

Again we look at the most critical two month – October and November

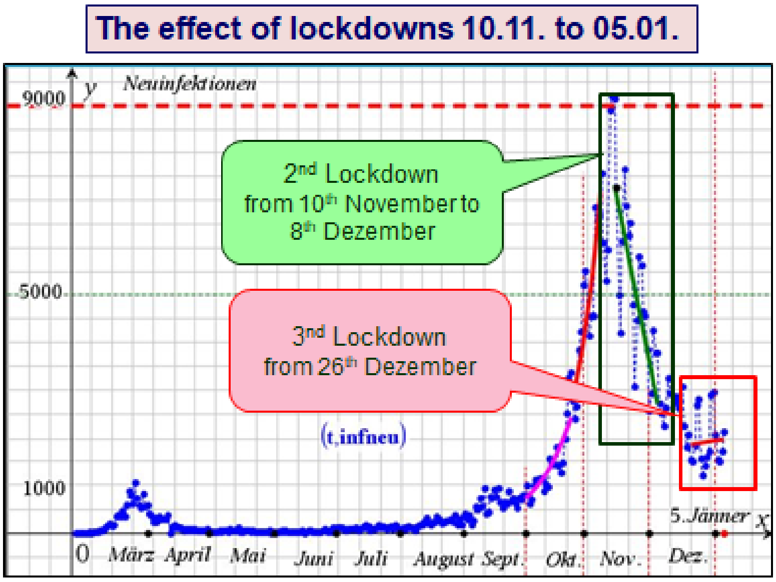


01.10. – 22.10: Exponential growth $fexpokt1(x) := 0{,}00296 \cdot 1{,}0579^{x}$

23.10. – 07.11.: Exponential growth $fexpokt2(x) := 0{,}000113 \cdot 1{,}0726^{x}$

The exponential regression functions "fexpokt1" and "fexpoct2" show the growing growth factor from beginning of October to November. Again a confirmation of the assumption that in this period we had in Austria an over-exponential growth of infected persons

Task 3: The development of the infection in the two months caused the decision of further lockdowns:



freglock2(x):=-220·x+65077

freglock3(x):=9·x-498

The 2nd lockdown at first seemed to be successful the tendency was falling. But after 3 weeks the number of new infections per day was still about 3000. This is much too high to allow relaxations.

Therefore the government decided a 3rd lockdown beginning at Christmas. This lockdown showed no effect. The number of infected persons was even increasing. So we are now living in the 4th lockdown not knowing how long it will last.

Task 4: Calculating the double-time d for exponential periods.

This important parameter for interpreting the consequence of such an exponential growth could be calculated by using the „p time d"-rule

„p times d – rule: For exponential growth with the percentage p and double–time d it applies: **p·d≈70**

$fexpokt1(x) := 0{,}00296 \cdot 1{,}0579^x$ ⇨ p1 = 5,79% ⇨ d1 ≈ 12 days

$fexpokt2(x) := 0{,}000113 \cdot 1{,}0726^x$ ⇨ p2 = 7,26% ⇨ d1 ≈ 9,6 days

The dangerous double time of beginning of October had even become shorter – another demonstration of an over-exponential growth.

Didactic comment to part 2:

To get a better understanding or finding a solution of the applied problem is one goal but as important is to question the mathematical activity. It is not enough to use technology as a black box for calculating and for experimenting. More than ever an exactifying learning phase is necessary.

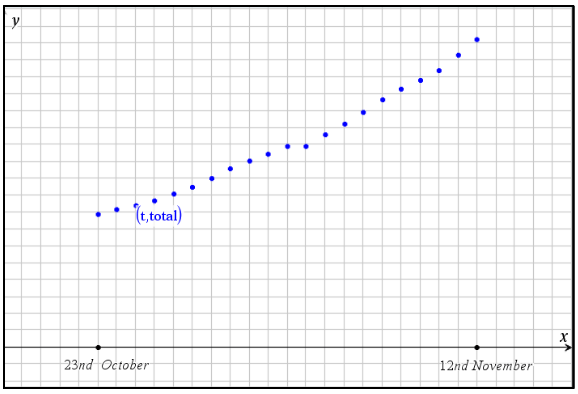
The experimental learning phase often at first possible when using technology must not replace the exactifying phase, it must prepare it or show the necessity.

Part 2: What can we learn from Corona about mathematics?

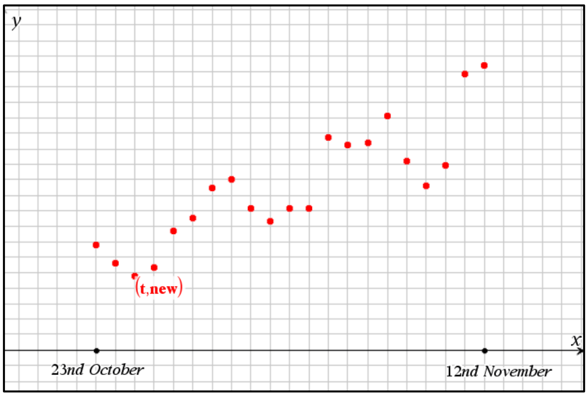
Task 5: What is the mathematical relationship between the total number of infected persons and the number of the daily new infected persons? Investigate this relationship in a period of exponential growth (e.g. end of October, beginning of November)

We draw the graphs of the total number of infected persons and the number of the daily new infected persons in the time interval from October 23rd to November 12th. We use the data of the given tables.

Totak number of infected persons



Daily new infected persons



In part 1 we have found the assumption that

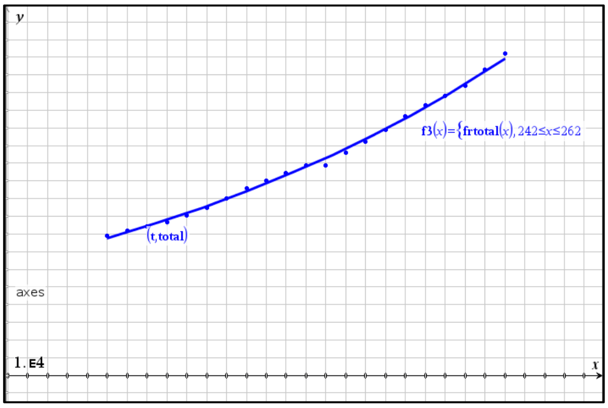
The fluctuation is less explainable by the in this period it was an exponential growth

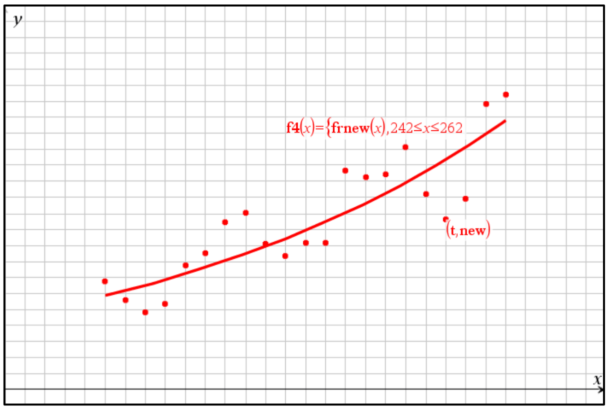
evolution of the infection, but by the irregular reporting of data from the regions

We can calculate the exponential regression functions and draw their graphs

$\text{frtotal}(x) = 3.0694 \cdot 1.04278^{x}$

$\text{frnew}(x) = 0.0083 \cdot 1.05420^{x}$



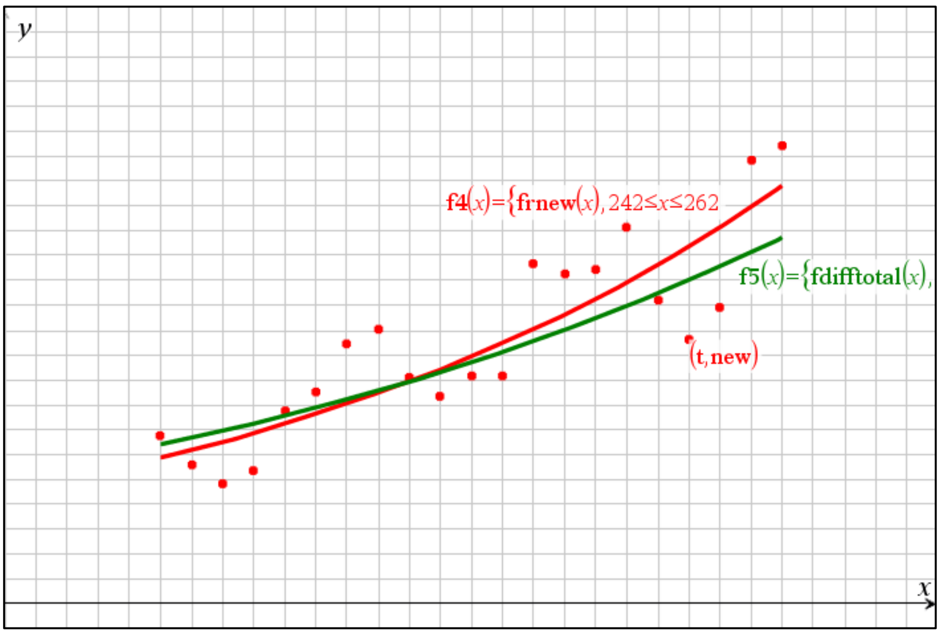


The mathematical relationship between the total number of infected persons and the number of the daily new infected persons

Interpretation 1:

We can get the number of the daily new infected persons by forming the difference of the total number of infected persons on two consecutive days:

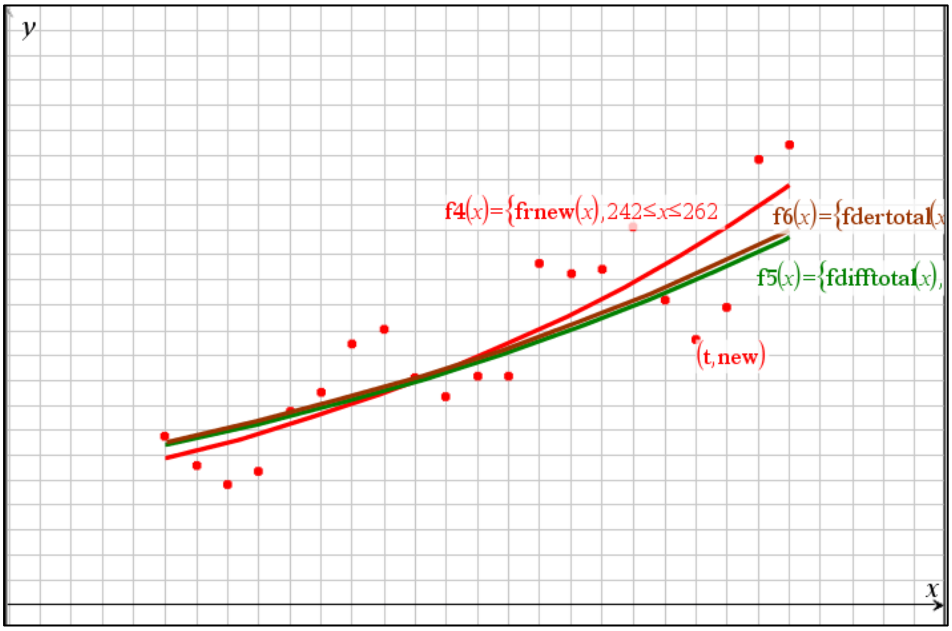
$\text{fdifftotal}(x) = \text{frtotal}(x) - \text{frtotal}(x-1) = 0.1259 \cdot 1.04278^{x}$



Interpretation 2:

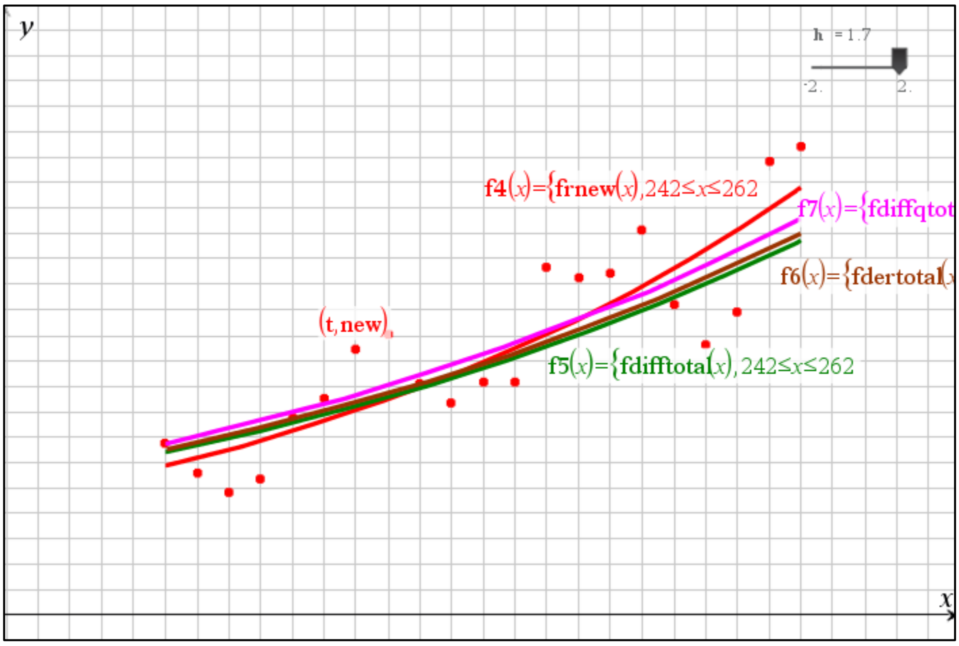
But the difference of the total number of infected persons on two consecutive days is the difference quotient or the mean rate of change $\frac{f(x+h)-f(x)}{h}$ for h = 1.

The derivative of the function of the total number of infected persons is the limit of the difference quotient: $\text{fdertotal}(x) = \frac{d}{dx}(\text{frtotal}(x)) = 0.12859 \cdot 1.04278^x$



Interpretation 3:

We define the difference quotient function $\text{fdiffqtotal}(x) = \frac{\text{frtotal}(x+h) - \text{frtotal}(x)}{h}$ and define a slider for h with $h \in [-2;+2]$:



For $h \to 0$ the graph of the difference quotient function moves against the graph of the derivative, for h = 0 the graph disappears.

Task 6: Prove the rules which you have used in Part 1.

Proof 1:

Proposition 1: Given is an exponential function with the function equation $y = c \cdot a^x$.

Claim: When using a logarithmic scale for the function values the graph is a straight line.

Proof: $y = c \cdot a^x$

$\ln(y) = \ln(c) + x \cdot \ln(a)$

If we designate $\ln(y) = z$ and $\ln(c) = s$ and $\ln(a) = r$

$z = r \cdot x + s$ The function equation of a straight line z = f(x)

Proof 2:

Proposition 1: „p times d – rule:

For exponential growth with the percentage p and double –time d it applies:

$$\mathbf{p \cdot d \approx 70}$$

Given is a capital k with a yearly percentage p

After one year:

- The capital for yearly interest $k_1 = k \cdot \left(1 + \frac{p}{100}\right) \Rightarrow k_1 = k \cdot (1+\lambda)$ for $\lambda = \frac{p}{100}$
- The capital for a six-monthly interest $k_{1/2} = k \cdot \left(1 + \frac{p}{2 \cdot 100}\right)^2 = k \cdot \left(1 + \frac{\lambda}{2}\right)^2$
- The capital for a monthly interest $k_{1/12}1 = k \cdot \left(1 + \frac{p}{12 \cdot 100}\right)^{12} = k \cdot \left(1 + \frac{\lambda}{12}\right)^{12}$
- The capital with ab interest after one n[th] of the year $k_{1/n} = k \cdot \left(1 + \frac{p}{n \cdot 100}\right)^n = k \cdot \left(1 + \frac{\lambda}{n}\right)^n = k \cdot \left(\left(1 + \frac{1}{\frac{n}{\lambda}}\right)^{\frac{n}{\lambda}}\right)^{\lambda}$

- The capital with an interest after one n^{th} of the year $\quad k_{1/n} = k \cdot \left(\left(1 + \frac{1}{m} \right)^m \right)^\lambda$ for $m = \frac{n}{\lambda}$
- The capital with an instantaneous interest $n \to \infty$ $\quad k_\infty = k \cdot \lim_{m \to \infty} \left(\left(1 + \frac{1}{m} \right)^m \right)^\lambda = k \cdot e^\lambda$

The capital as a function with respect to the time t: $k(t) = k \cdot e^{\lambda \cdot t}$

After which time d the capital is twice as large?

$$2 = e^{\lambda \cdot d}$$

$$\ln(2) = \lambda \cdot d$$

$$\ln(2) = \frac{p}{100} \cdot d$$

$$\mathbf{d \cdot p = 100 \cdot \ln(2) \approx 70}$$

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